PPT discrimination of twisted Bell states with a partially entangled resource

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Abstract

We determine the optimal probability of correctly discriminating the four orthogonal two-qubit states $\alpha|00\rangle + \beta|11\rangle$, $\beta|00\rangle - \alpha|11\rangle$, $\alpha|01\rangle + \beta|10\rangle$ and $\beta|01\rangle - \alpha|10\rangle$, given with uniform probabilities together with an ancillary pair of qubits in the state $\sqrt{(1+\varepsilon)/2}|00\rangle + \sqrt{(1-\varepsilon)/2}|11\rangle$, by measurements whose elements have positive partial transpose, using a matching pair of primal and dual solutions of the associated semidefinite program. Specific results that we prove include: an exact formula, $\frac{1}{2}\big(1+\sqrt{1-v^2}\big)$ with $v = \max\{0, \alpha\beta(1+\varepsilon) - (1-\varepsilon)/2\}$, for this optimal probability; an explicit PPT measurement and an explicit dual witness in closed form whose values coincide; a characterization of the pairs (α, ε) for which the four states are perfectly discriminated, namely $2\alpha\beta(1+\varepsilon) \leq 1-\varepsilon$; and the observation that the teleportation protocol that is optimal for Bell states is strictly suboptimal whenever $0 < \beta < \alpha$ and $0 < \varepsilon < 1$.

1 Introduction

A measurement performed by two parties on a bipartite system is said to be a *PPT measurement* when each of its elements has a positive semidefinite partial transpose with respect to the bipartition between the parties. The class of PPT measurements contains the class of separable measurements, and therefore every measurement that can be implemented by local operations and classical communication (LOCC); unlike those two classes it is defined by linear and positive semidefinite constraints alone, so that the best PPT measurement for a given task is the solution of a semidefinite program. This paradigm serves as the principal tool through which limitations on local state discrimination have been established in closed form, and it is the paradigm of the present paper.

State discrimination with an entangled resource

In the state discrimination problems considered here, two parties, Alice and Bob, share a system in one of N known orthogonal pure states $\rho_1, \dots, \rho_N$, each with probability $1/N$, and must identify which one they hold. A measurement is a collection $\{P_1, \dots, P_N\}$ of positive semidefinite operators summing to the identity, and its success probability is

$$\frac{1}{N}\sum_{k=1}^{N}\langle P_k, \rho_k\rangle, \tag{1}$$

where $\langle A, B\rangle = \mathrm{Tr}(A^*B)$ denotes the Hilbert–Schmidt inner product. A measurement whose elements are all separable across the bipartition is a separable measurement, and a measurement that can be carried out by an LOCC protocol is an LOCC measurement; the three classes are nested, LOCC measurements within separable measurements within PPT measurements, so that an upper bound on the success probability of PPT measurements applies to all three, while a success probability attained by an LOCC protocol is a lower bound for all three. When the states are entangled, the parties may fail to discriminate them perfectly by any LOCC measurement even though a global measurement succeeds with certainty, and one way to quantify this failure is to give the parties an additional entangled pair as a resource and to ask how the optimal success probability grows with the entanglement of the resource.

The four Bell states

$$\begin{aligned} |\phi_1\rangle &= \tfrac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \quad |\phi_2\rangle = \tfrac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big), \\ |\phi_3\rangle &= \tfrac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \quad |\phi_4\rangle = \tfrac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big), \end{aligned} \tag{2}$$

are the standard example. Ghosh, Kar, Roy, Sen(De) and Sen [GKR+01] proved that not even three of them can be perfectly discriminated by LOCC, and it follows from a result of Nathanson [Nat04] that four equiprobable Bell states are discriminated by LOCC with probability at most $1/2$, which is attained by measuring both qubits in the computational basis. The same value $1/2$ is optimal for PPT measurements: the operator $Y = \mathbb{1}/8$ on the two signal qubits, with $Q_i = \mathrm{T}(Y - \rho_i/4)$, is feasible for the dual of the semidefinite program that Cosentino [Cos12] introduced for PPT discrimination, an argument that reappears as the corner $\alpha = \beta$, $\varepsilon = 1$ of Theorem 14 below. Bandyopadhyay, Cosentino, Johnston, Russo, Watrous and Yu [BCJ+14] then determined the effect of an ancillary pair of qubits in the state

$$|\tau_\varepsilon\rangle = \sqrt{\tfrac{1+\varepsilon}{2}}\,|00\rangle + \sqrt{\tfrac{1-\varepsilon}{2}}\,|11\rangle, \qquad \varepsilon \in [0,1], \tag{3}$$

which is maximally entangled when $\varepsilon = 0$ and a product state when $\varepsilon = 1$: for the four Bell states the optimal probability of PPT measurements is exactly $\frac{1}{2}\big(1 + \sqrt{1-\varepsilon^2}\big)$, and it is attained by the LOCC protocol in which Alice teleports her qubit through the resource and Bob measures, so that the LOCC, separable and PPT values coincide. For three Bell states the same paper proved that the optimal separable probability is $\frac{1}{3}\big(2 + \sqrt{1-\varepsilon^2}\big)$, whereas Yu, Duan and Ying [YDY12] had shown that PPT measurements discriminate three Bell states perfectly if and only if $\varepsilon \le 1/3$; the two classes therefore differ for three states and agree for four. Further results on local discrimination with and without resources, including the general facts that a complete orthonormal basis is perfectly LOCC-discriminated only if all of its members are product states and that a maximally entangled qubit pair is a universal resource for discriminating a basis of two qubits, may be found in [WH02, HDSH03, DFXY07, BGK11, YDY11, CR13, YO15, BHN15, BHN17].

A twisted Bell basis

The present paper concerns the *twisted Bell basis*, the orthonormal basis of two qubits obtained by replacing the equal Schmidt coefficients of the Bell states by an arbitrary pair; its members are the *twisted Bell states*. Let $\alpha \ge \beta \ge 0$ be real numbers with $\alpha^2 + \beta^2 = 1$, and define

$$\begin{aligned} |\psi_0\rangle &= \alpha|00\rangle + \beta|11\rangle, \quad |\psi_1\rangle = \beta|00\rangle - \alpha|11\rangle, \\ |\psi_2\rangle &= \alpha|01\rangle + \beta|10\rangle, \quad |\psi_3\rangle = \beta|01\rangle - \alpha|10\rangle. \end{aligned} \tag{4}$$

Each of these states has concurrence $c = 2\alpha\beta$; they are the Bell states when $\alpha = \beta = 1/\sqrt{2}$ and the computational product basis when $\alpha = 1$. The task is to discriminate the four states $|\psi_i\rangle \otimes |\tau_\varepsilon\rangle$, given with uniform probabilities, by a PPT measurement with respect to the bipartition between Alice, who holds the first qubit of each pair, and Bob, who holds the second, and the quantity of interest is the optimal success probability $\operatorname{opt}(\alpha, \varepsilon)$ as a function of the two parameters. Three sides of the parameter square are elementary. When $\varepsilon = 0$ the resource is maximally entangled and teleportation gives $\operatorname{opt}(\alpha, 0) = 1$ for every α; when $\alpha = 1$ the basis is a product basis and $\operatorname{opt}(1, \varepsilon) = 1$ for every ε; and when $\alpha = \beta$ the value $\frac{1}{2}(1 + \sqrt{1 - \varepsilon^2})$ is the theorem of [BCJ+14] quoted above. Bandyopadhyay, Halder and Nathanson [BHN17] list the discrimination of bases of this kind, even without a resource and even for LOCC, among the open problems of the subject, and to our knowledge no value in the interior of the square was known.

A natural guess is available. If Alice measures her two qubits in the Bell basis, thereby teleporting her half of the signal state to Bob through the imperfect resource, and Bob then performs the best measurement on his two qubits, the success probability is $\frac{1}{2}(1 + \sqrt{1 - (c\varepsilon)^2})$ (Proposition 17), which agrees with all three known sides of the square. This guess is wrong in the interior. The optimal PPT measurement does strictly better whenever $0 < \beta < \alpha$ and $0 < \varepsilon < 1$, and there is a whole region of the square, bounded by the curve $\varepsilon = (1 - c)/(1 + c)$, in which the four states are perfectly discriminated although the resource is only partially entangled.

The method

The value is obtained from a matching pair of solutions of the semidefinite program that defines it: an explicit PPT measurement whose success probability equals $\frac{1}{2}(1 + \sqrt{1 - v^2})$, and an explicit dual witness whose trace equals the same number. Both are found through the symmetry of the ensemble. The four states (4) are the orbit of $|\psi_0\rangle$ under a group of eight local unitary operators, taken modulo phases, that contains the parity operator $\sigma_z \otimes \sigma_z$, which fixes every $|\psi_i\rangle$ up to sign, and the resource (3) is fixed by every local phase rotation $e^{i\varphi\sigma_z} \otimes e^{-i\varphi\sigma_z}$; since local unitary operators preserve the PPT property, it is natural to look for a measurement that is covariant under this group and block diagonal with respect to the parity of the signal pair and the charge of the resource pair, and the measurement constructed in Section 3 is of this form. Within that block structure the measurement is built from a single vector w on a four-dimensional subspace, together with padding blocks that make it complete and PPT, and its feasibility reduces to three inequalities on the two components of w. Maximizing the success probability over this family is a one-variable problem whose solution is $v = c\lambda_0 - \lambda_1$, with $\lambda_0 = (1 + \varepsilon)/2$ and $\lambda_1 = (1 - \varepsilon)/2$ the Schmidt weights of the resource, and this is where the formula comes from. The dual witness is then determined by complementary slackness: its four free parameters are the unique solution of the linear system expressing that the dual slack annihilates the support of the optimal measurement, and its feasibility is proved through one lemma on 4×4 matrices of the form $\mathbb{1} \otimes K - f\,\sigma_y \otimes \sigma_y$ and a rank-two argument whose two invariants are computed on the page.

Overview of results

Our results concern PPT measurements and are based on semidefinite programming duality; the notion of a matching pair, a PPT measurement and a dual witness that are both displayed in closed form, plays the central role. The symmetry of the ensemble is what finds the pair, and in the proofs it enters only through two displayed conjugation identities, (14) and (39), which reduce the four primal values and the four dual constraints to one each; no averaging argument is invoked to show that an optimal measurement can be taken covariant. The following specific results are among those we prove:

- An exact formula for the optimal probability with which the states $|\psi_i\rangle \otimes |\tau_\varepsilon\rangle$ are discriminated by PPT measurements, for every $\alpha \geq \beta \geq 0$ and every $\varepsilon \in [0,1]$: it equals $\frac{1}{2}\big(1 + \sqrt{1-v^2}\big)$ with $v = \max\{0, 2\alpha\beta\lambda_0 - \lambda_1\}$. On the Bell line $\alpha = \beta$ this is the formula of [BCJ+14], and for a product resource it gives the value α^2 of the local measurement in the computational basis. The formula answers, for PPT measurements and for bases in which the two pairs of states share their Schmidt coefficients, the question raised for bases of this kind in [BHN17].
- A characterization of perfect discrimination: the four states are perfectly discriminated by a PPT measurement if and only if $2\alpha\beta\lambda_0 \leq \lambda_1$, that is, if and only if the concurrence of the signal states is at most the ratio of the smaller to the larger Schmidt weight of the resource. This answers, for the twisted Bell basis and a qubit resource, the question of how much entanglement is needed to discriminate a basis of two qubits perfectly, which [BCJ+14] raise in their concluding section for maximally entangled states; on the Bell line it recovers the result of [YDY12] that only a maximally entangled resource allows the four Bell states to be perfectly discriminated.
- An explicit family of PPT measurements, parametrized by two vectors in $\mathbb{R}^2$, whose feasibility is characterized by three inequalities and whose best member attains the optimal value; and an explicit dual witness in closed form, with its feasibility proved constraint by constraint. The measurement is not a teleportation protocol: the teleportation protocol that is optimal on the Bell line is strictly suboptimal whenever $0 < \beta < \alpha$ and $0 < \varepsilon < 1$, so the coincidence of the PPT value with the teleportation value that [BCJ+14] established for Bell states does not extend to the twisted Bell basis, and the question of which LOCC protocol is optimal there is left open.

2 The semidefinite program and its dual

This section fixes the notation, displays the semidefinite program whose value is $\mathrm{opt}(\alpha, \varepsilon)$ together with its dual, and proves the three general facts on which every later proof rests.

The program

Let $\mathcal{X}_1$, $\mathcal{Y}_1$, $\mathcal{X}_2$ and $\mathcal{Y}_2$ be copies of $\mathbb{C}^2$ with computational basis $\{|0\rangle, |1\rangle\}$, let $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$ be Alice's system and $\mathcal{Y} = \mathcal{Y}_1 \otimes \mathcal{Y}_2$ be Bob's, and write every operator on the tensor product $\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2$ in that order, so that a basis vector $|x_1 y_1 x_2 y_2\rangle$ has x_1, x_2 on Alice's side. The partial transpose $\mathrm{T}_\mathcal{Y}$ is the linear map that transposes Bob's two indices in the computational basis,

$$\mathrm{T}_\mathcal{Y}\big(|x_1 y_1 x_2 y_2\rangle\langle x_1' y_1' x_2' y_2'|\big) = |x_1 y_1' x_2 y_2'\rangle\langle x_1' y_1 x_2' y_2|; \tag{5}$$

it is the composition $\mathrm{T}_{\mathcal{Y}_1} \circ \mathrm{T}_{\mathcal{Y}_2}$ of the transposes of the single qubits $\mathcal{Y}_1$ and $\mathcal{Y}_2$, it preserves the trace and Hermiticity, it is an involution, it is self-adjoint with respect to the Hilbert–Schmidt inner product, and for operators U on $\mathcal{X}$ and V on $\mathcal{Y}$ it satisfies

$$\mathrm{T}_\mathcal{Y}\big((U \otimes V) P (U \otimes V)^*\big) = (U \otimes \overline{V})\ \mathrm{T}_\mathcal{Y}(P)\ (U \otimes \overline{V})^*, \tag{6}$$

where $\overline{V}$ is the entrywise complex conjugate of V. An operator P is PPT when $P \succeq 0$ and $\mathrm{T}_{\mathcal{Y}}(P) \succeq 0$; since transposing Alice's indices instead gives the transpose of $\mathrm{T}_{\mathcal{Y}}(P)$, the property does not depend on the side. Hereafter $|\psi_0\rangle, \ldots, |\psi_3\rangle$ are the states (4) on $\mathcal{X}_1 \otimes \mathcal{Y}_1$, $|\tau\rangle = |\tau_\varepsilon\rangle$ is the state (3) on $\mathcal{X}_2 \otimes \mathcal{Y}_2$, and

$$\rho_i = \big(|\psi_i\rangle \otimes |\tau\rangle\big)\big(\langle\psi_i| \otimes \langle\tau|\big), \qquad i = 0, 1, 2, 3. \tag{7}$$

The optimal PPT success probability is the value of the primal problem below, and the dual problem is obtained from it by the usual Lagrangian computation, with a Hermitian multiplier Y for the completeness constraint and positive semidefinite multipliers Q_i for the partial-transpose constraints.

$$\underline{\text{Primal problem}}$$
$$\begin{aligned} \text{maximize:}\quad & \tfrac{1}{4}\sum_{i=0}^{3}\langle P_i, \rho_i\rangle \\ \text{subject to:}\quad & P_0 + P_1 + P_2 + P_3 = \mathbb{1}, \\ & P_i \succeq 0,\ \ \mathrm{T}_{\mathcal{Y}}(P_i) \succeq 0 \quad (i = 0, \ldots, 3). \end{aligned}$$

$$\underline{\text{Dual problem}}$$
$$\begin{aligned} \text{minimize:}\quad & \mathrm{Tr}(Y) \\ \text{subject to:}\quad & Y - \tfrac{1}{4}\rho_i - \mathrm{T}_{\mathcal{Y}}(Q_i) \succeq 0 \quad (i = 0, \ldots, 3), \\ & Q_i \succeq 0,\ \ Y \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y}). \end{aligned}$$

The primal feasible set is compact and nonempty, so the maximum exists, and it is denoted $\mathrm{opt}(\alpha, \varepsilon)$.

Proposition 1 (Weak duality)**.** *Let $(P_0, \ldots, P_3)$ be feasible for the primal problem and let $(Y, Q_0, \ldots, Q_3)$ be feasible for the dual problem. It holds that $\frac{1}{4}\sum_i \langle P_i, \rho_i\rangle \le \mathrm{Tr}(Y)$.*

Proof. The proof is a chain of two inequalities. Since $\sum_i P_i = \mathbb{1}$, one has $\mathrm{Tr}(Y) = \sum_i \langle P_i, Y\rangle$, and since $P_i \succeq 0$ and $Y - \frac{1}{4}\rho_i - \mathrm{T}_{\mathcal{Y}}(Q_i) \succeq 0$, the inner product of the two is nonnegative, so that

$$\mathrm{Tr}(Y) \ge \sum_{i=0}^{3}\Big\langle P_i, \tfrac{1}{4}\rho_i + \mathrm{T}_{\mathcal{Y}}(Q_i)\Big\rangle = \frac{1}{4}\sum_{i=0}^{3}\langle P_i, \rho_i\rangle + \sum_{i=0}^{3}\langle \mathrm{T}_{\mathcal{Y}}(P_i), Q_i\rangle, \tag{8}$$

where the self-adjointness of $\mathrm{T}_{\mathcal{Y}}$ was used. Each term of the last sum is the inner product of two positive semidefinite operators and is therefore nonnegative, which establishes the inequality. □

Thus every dual feasible pair provides an upper bound of $\mathrm{Tr}(Y)$ on the success probability of every PPT measurement, and every PPT measurement provides a lower bound on $\mathrm{Tr}(Y)$ for every dual feasible pair. The program also satisfies strong duality, because $P_i = \mathbb{1}/4$ is strictly feasible, but this fact is not needed: the theorem is proved by exhibiting a PPT measurement and a dual witness with the same value.

Notation for the twisted Bell basis

The parameters of the problem are collected as follows. Let $c = 2\alpha\beta \in [0, 1]$ be the concurrence of the signal states and $\sigma = \alpha^2 - \beta^2 = \sqrt{1-c^2}$, let $\lambda_0 = (1+\varepsilon)/2$ and $\lambda_1 = (1-\varepsilon)/2$ be the Schmidt weights of the resource, so that $\lambda_0 + \lambda_1 = 1$ and $\lambda_0 - \lambda_1 = \varepsilon$, and define

$$v = c\lambda_0 - \lambda_1, \qquad s = \sqrt{1-v^2} \quad (\text{when } v \geq 0). \tag{9}$$

Four identities, each a one-line consequence of $\lambda_0 + \lambda_1 = 1$, $\lambda_0 - \lambda_1 = \varepsilon$ and $(\alpha \pm \beta)^2 = 1 \pm c$, are used throughout:

$$\lambda_0(1+c) = 1+v, \qquad \lambda_0(1-c) = \varepsilon - v, \qquad (\varepsilon - v) + 2\lambda_1 = 1 - v, \qquad \sqrt{\lambda_0}\,(\alpha \pm \beta) = \sqrt{1+v},\ \sqrt{\varepsilon - v}. \tag{10}$$

Since $c \leq 1$, the second identity gives $v \leq \varepsilon \leq 1$, and the third gives $1 - v \geq 2\lambda_1 \geq 0$, with $v = 1$ only when $\lambda_1 = 0$ and $c = 1$. For $0 \leq v < 1$ put

$$r = \sqrt{\frac{\varepsilon - v}{1-v}} \in [0, 1], \qquad \kappa = \sqrt{\frac{\lambda_1}{1-v}} \in \left[0, \tfrac{1}{\sqrt{2}}\right], \qquad r^2 + 2\kappa^2 = 1, \tag{11}$$

the last equality being the third identity of (10).

The signal space decomposes by parity as $\mathcal{X}_1 \otimes \mathcal{Y}_1 = \mathcal{E}_+ \oplus \mathcal{E}_-$ with $\mathcal{E}_+ = \operatorname{span}\{|00\rangle, |11\rangle\}$ and $\mathcal{E}_- = \operatorname{span}\{|01\rangle, |10\rangle\}$, the eigenspaces of $\sigma_z \otimes \sigma_z$, and the resource space decomposes by charge as $\mathcal{X}_2 \otimes \mathcal{Y}_2 = \mathcal{S} \oplus \mathcal{C}_+ \oplus \mathcal{C}_-$ with $\mathcal{S} = \operatorname{span}\{|00\rangle, |11\rangle\}$, $\mathcal{C}_+ = \operatorname{span}\{|01\rangle\}$ and $\mathcal{C}_- = \operatorname{span}\{|10\rangle\}$, the eigenspaces of the phase rotations $e^{i\varphi\sigma_z} \otimes e^{-i\varphi\sigma_z}$. Each of $\mathcal{E}_+$, $\mathcal{E}_-$ and $\mathcal{S}$ is identified with $\mathbb{C}^2$ through the ordered bases just listed, and in these coordinates

$$|\psi_0\rangle \mapsto a = (\alpha, \beta), \quad |\psi_1\rangle \mapsto a' = (\beta, -\alpha) \text{ in } \mathcal{E}_+, \qquad |\psi_2\rangle \mapsto a, \quad |\psi_3\rangle \mapsto a' \text{ in } \mathcal{E}_-, \qquad |\tau\rangle \mapsto t = (\sqrt{\lambda_0}, \sqrt{\lambda_1}) \text{ i} \tag{12}$$

so that $\{a, a'\}$ is an orthonormal basis of $\mathbb{R}^2$ and ρ_0 is supported on $\mathcal{E}_+ \otimes \mathcal{S}$, where it equals $|a\rangle\langle a| \otimes |t\rangle\langle t|$. The Pauli operators are σ_x, σ_y, σ_z; further $J = i\sigma_y = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so that J is real, $J^\intercal = -J$, $Ja = a'$ and $Ja' = -a$; and $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$ on $\mathcal{S}$. A vector $u \in \mathcal{E}_+ \otimes \mathcal{S}$ is written either as the flat 4-vector of its coefficients in the order $|00\rangle|00\rangle$, $|00\rangle|11\rangle$, $|11\rangle|00\rangle$, $|11\rangle|11\rangle$ (signal pair first, then resource pair), or as the 2×2 array U with U_{jk} the coefficient of the jth basis vector of $\mathcal{E}_+$ tensored with the kth of $\mathcal{S}$; in the array convention $(A \otimes B)u$ corresponds to $AUB^\intercal$, and the array with columns $x, y \in \mathbb{R}^2$ is written $[x, y]$.

Finally, with $X = \sigma_x$ and $Z = \sigma_z$, define on $\mathcal{X}_1 \otimes \mathcal{Y}_1$ the unitary operators

$$g_0 = \mathbb{1} \otimes \mathbb{1}, \qquad g_1 = ZX \otimes X, \qquad g_2 = \mathbb{1} \otimes X, \qquad g_3 = ZX \otimes \mathbb{1}. \tag{13}$$

Since $ZX|0\rangle = -|1\rangle$ and $ZX|1\rangle = |0\rangle$, direct substitution in (4) gives $g_i|\psi_0\rangle = |\psi_i\rangle$ for each i, hence

$$\rho_i = (g_i \otimes \mathbb{1})\, \rho_0\, (g_i \otimes \mathbb{1})^*, \qquad i = 0, 1, 2, 3. \tag{14}$$

Together with $h = Z \otimes Z$, which satisfies $h|\psi_i\rangle = \pm|\psi_i\rangle$, the operators g_1, g_2, g_3 generate, modulo phases, an abelian group G of order eight whose action on the four states is transitive with stabilizer $\{\mathbb{1}, h\}$; and each element of G is a local unitary operator whose factor on Bob's side is real, so that by (6) conjugation by $g \otimes \mathbb{1}$, $g \in G$, commutes with $\mathrm{T}_{\mathcal{Y}}$ and preserves the PPT property. In the coordinates of (12), g_1 acts on $\mathcal{E}_+$ and on $\mathcal{E}_-$ as J, while g_2 maps $\mathcal{E}_-$ onto $\mathcal{E}_+$ and

$\mathcal{E}_+$ onto $\mathcal{E}_-$ as the identity of $\mathbb{C}^2$, and $g_3 = g_1 g_2$. The commutant of G on $\mathcal{X}_1 \otimes \mathcal{Y}_1$ is spanned by $\mathbb{1} \otimes \mathbb{1}$ and $\sigma_y \otimes \sigma_x$: conjugation by a Pauli product maps every Pauli product to plus or minus itself, so an operator commutes with G if and only if each Pauli product in its expansion does, and among the sixteen only these two commute with $\mathbb{1} \otimes X$, $ZX \otimes \mathbb{1}$ and $Z \otimes Z$; and $\sigma_y \otimes \sigma_x$ acts as σ_y on $\mathcal{E}_+$ and as σ_y on $\mathcal{E}_-$ in the coordinates of (12).

Partial transposes and a lemma on 4×4 matrices

Lemma 2. *Let $|\phi\rangle = \sum_{j,k} M_{jk} |j\rangle_{\mathcal{A}} |k\rangle_{\mathcal{B}}$ be a vector on $\mathcal{A} \otimes \mathcal{B}$ with $\mathcal{A} \cong \mathcal{B} \cong \mathbb{C}^n$, and let F be the swap operator of $\mathcal{A}$ and $\mathcal{B}$. Then $\mathrm{T}_{\mathcal{B}}(|\phi\rangle\langle\phi|) = (M \otimes M^*)F$, where M^* is the adjoint of M. In particular, for $M = \mathrm{diag}(m_0, \ldots, m_{n-1})$ real, the partial transpose of $|\phi\rangle\langle\phi|$ has eigenvalue m_j^2 on $|jj\rangle$ and eigenvalues $\pm m_j m_k$ on $(|jk\rangle \pm |kj\rangle)/\sqrt{2}$ for $j < k$; consequently*

$$\begin{aligned} \mathrm{T}_{\mathcal{Y}_1}\big(|\psi_0\rangle\langle\psi_0|\big) &= \alpha^2 |00\rangle\langle 00| + \beta^2 |11\rangle\langle 11| + \alpha\beta\big(|01\rangle\langle 10| + |10\rangle\langle 01|\big), \\ \mathrm{T}_{\mathcal{Y}_2}\big(|\tau\rangle\langle\tau|\big) &= \lambda_0 |00\rangle\langle 00| + \lambda_1 |11\rangle\langle 11| + \sqrt{\lambda_0\lambda_1}\big(|01\rangle\langle 10| + |10\rangle\langle 01|\big), \end{aligned} \tag{15}$$

with eigenvalues $\alpha^2, \beta^2, \pm\alpha\beta$ and $\lambda_0, \lambda_1, \pm\sqrt{\lambda_0\lambda_1}$ respectively, and $\mathrm{T}_{\mathcal{Y}}(\rho_0)$ is the tensor product of the two. Moreover, for the singlet $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ of $\mathcal{X}_1 \otimes \mathcal{Y}_1$,

$$\mathrm{T}_{\mathcal{Y}_1}\big(|\Psi^-\rangle\langle\Psi^-|\big) = \tfrac{1}{2}\mathbb{1} - |\Phi^+\rangle\langle\Phi^+|, \tag{16}$$

which restricted to $\mathcal{E}_+$ equals $-\frac{1}{2}\sigma_x$ in the coordinates of (12) and restricted to $\mathcal{E}_-$ equals $\frac{1}{2}\mathbb{1}$.

Proof. The general formula is obtained by writing $|\phi\rangle\langle\phi|$ through the swap operator, and the four special cases follow by substitution. Writing $|\Omega\rangle = \sum_k |kk\rangle$, one has $|\phi\rangle = (M \otimes \mathbb{1})|\Omega\rangle$, $\mathrm{T}_{\mathcal{B}}(|\Omega\rangle\langle\Omega|) = F$, and $\mathrm{T}_{\mathcal{B}}$ commutes with conjugation by operators on $\mathcal{A}$, so that $\mathrm{T}_{\mathcal{B}}(|\phi\rangle\langle\phi|) = (M \otimes \mathbb{1})F(M^* \otimes \mathbb{1}) = (M \otimes \mathbb{1})(\mathbb{1} \otimes M^*)F = (M \otimes M^*)F$, using $F(M^* \otimes \mathbb{1}) = (\mathbb{1} \otimes M^*)F$. For diagonal real M the operator $(M \otimes M)F$ maps $|jk\rangle$ to $m_j m_k |kj\rangle$, which gives the stated eigenvectors and eigenvalues; (15) is the case $n = 2$ with $M = \mathrm{diag}(\alpha, \beta)$ and $M = \mathrm{diag}(\sqrt{\lambda_0}, \sqrt{\lambda_1})$, and $\mathrm{T}_{\mathcal{Y}} = \mathrm{T}_{\mathcal{Y}_1} \circ \mathrm{T}_{\mathcal{Y}_2}$ acts factorwise on $\rho_0 = |\psi_0\rangle\langle\psi_0| \otimes |\tau\rangle\langle\tau|$. For the singlet, $M = \frac{1}{\sqrt{2}}J$, and since J is real and antisymmetric its adjoint is $M^* = M^\mathsf{T} = -M$, so $M \otimes M^* = -\frac{1}{2}J \otimes J$; the operator $-\frac{1}{2}(J \otimes J)F$ maps $|00\rangle \mapsto -\frac{1}{2}|11\rangle$, $|11\rangle \mapsto -\frac{1}{2}|00\rangle$, $|01\rangle \mapsto \frac{1}{2}|01\rangle$ and $|10\rangle \mapsto \frac{1}{2}|10\rangle$, which is (16), whose restrictions to $\mathcal{E}_+$ and $\mathcal{E}_-$ are read off from $|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}(\mathbb{1} + \sigma_x)$ on $\mathcal{E}_+$ in the coordinates of (12). ☐

Lemma 3. *Let K be a real symmetric 2×2 matrix with eigenvalues k_1, k_2, and let f be a real number. The real symmetric 4×4 matrix $\Omega = \mathbb{1} \otimes K - f\,\sigma_y \otimes \sigma_y$ has the eigenvalues*

$$\frac{k_1 + k_2}{2} \pm \sqrt{\left(\frac{k_1 - k_2}{2}\right)^2 + f^2}, \tag{17}$$

each of multiplicity two. Consequently $\Omega \succeq 0$ if and only if $K \succeq 0$ and $\det K \geq f^2$, and if $\det K = f^2$ and $\mathrm{Tr}\,K > 0$ then Ω has rank two with both nonzero eigenvalues equal to $\mathrm{Tr}\,K$.

Proof. It will first be shown that the eigenvalues of Ω do not depend on the eigenvectors of K. Let O be a real orthogonal matrix with $O^\mathsf{T} K O = \mathrm{diag}(k_1, k_2)$. For every real 2×2 matrix O one has $O^{\mathsf{T}}JO = (\det O)J$, because $u^{\mathsf{T}}Jv = u_0v_1 - u_1v_0$ is the determinant form and $(Ou)^{\mathsf{T}}J(Ov) = \det(Ou, Ov) = \det(O)\det(u,v)$; as $\sigma_y = -iJ$, conjugation by O maps σ_y to $(\det O)\sigma_y$, hence

$$(\mathbb{1}\otimes O)^{\mathsf{T}}\,\Omega\,(\mathbb{1}\otimes O) = \frac{k_1+k_2}{2}\,\mathbb{1} + \frac{k_1-k_2}{2}\,(\mathbb{1}\otimes\sigma_z) - f\det(O)\,(\sigma_y\otimes\sigma_y). \tag{18}$$

The operators $\mathbb{1}\otimes\sigma_z$ and $\sigma_y\otimes\sigma_y$ are Hermitian involutions that anticommute, since $\sigma_z\sigma_y = -\sigma_y\sigma_z$, so their combination $a(\mathbb{1}\otimes\sigma_z) + b(\sigma_y\otimes\sigma_y)$ squares to $(a^2+b^2)\mathbb{1}$ and has trace zero, and therefore has eigenvalues $\pm\sqrt{a^2+b^2}$, each twice. This gives (17), and $\Omega \succeq 0$ if and only if $\frac{k_1+k_2}{2} \geq \sqrt{(\frac{k_1-k_2}{2})^2 + f^2}$, which holds if and only if $k_1+k_2 \geq 0$ and $\frac{(k_1+k_2)^2-(k_1-k_2)^2}{4} = k_1k_2 \geq f^2$. The last assertion is the case of equality. □

Every proof below is an application of Proposition 1: a PPT measurement gives a lower bound on $\mathrm{opt}(\alpha,\varepsilon)$ equal to its success probability, and a dual feasible pair gives an upper bound equal to $\mathrm{Tr}(Y)$.

3 An explicit family of PPT measurements

The measurements of this section are covariant: they have the form $P_i = (g_i\otimes\mathbb{1})P_0(g_i\otimes\mathbb{1})^*$ for a single real symmetric operator P_0 that commutes with $h\otimes\mathbb{1}$ and with the phase rotations of the resource pair, and is therefore block diagonal with respect to the decomposition $(\mathcal{E}_+\oplus\mathcal{E}_-)\otimes(\mathcal{S}\oplus\mathcal{C}_+\oplus\mathcal{C}_-)$. Such a P_0 has a 4×4 block on $\mathcal{E}_+\otimes\mathcal{S}$, a 4×4 block on $\mathcal{E}_-\otimes\mathcal{S}$, and four 2×2 blocks on $\mathcal{E}_+\otimes\mathcal{C}_+$, $\mathcal{E}_+\otimes\mathcal{C}_-$, $\mathcal{E}_-\otimes\mathcal{C}_+$ and $\mathcal{E}_-\otimes\mathcal{C}_-$, and each of P^{++} and P^{--}, its restrictions to $\mathcal{E}_+\otimes(\mathcal{X}_2\otimes\mathcal{Y}_2)$ and $\mathcal{E}_-\otimes(\mathcal{X}_2\otimes\mathcal{Y}_2)$, is a 2×2 array of 4×4 blocks $P^{\pm\pm}_{jk}$ indexed by the coordinates of $\mathcal{E}_+$ or $\mathcal{E}_-$. The completeness constraint takes a simple form for such operators.

Lemma 4. *Let $P_0 = P^{++}\oplus P^{--}$ be real symmetric and block diagonal as above, and let $P_i = (g_i\otimes\mathbb{1})P_0(g_i\otimes\mathbb{1})^*$. Then $P_0+P_1+P_2+P_3 = \mathbb{1}$ if and only if*

$$\mathrm{Tr}_{\mathcal{E}_+}(P^{++}) + \mathrm{Tr}_{\mathcal{E}_-}(P^{--}) = \mathbb{1}_{\mathcal{X}_2\otimes\mathcal{Y}_2} \qquad \textit{and} \qquad \mathrm{Anti}\big(P^{++}_{01}\big) + \mathrm{Anti}\big(P^{--}_{01}\big) = 0, \tag{19}$$

where $\mathrm{Tr}_{\mathcal{E}_+}$ and $\mathrm{Tr}_{\mathcal{E}_-}$ are the partial traces over the signal sector and $\mathrm{Anti}(M) = \frac{1}{2}(M - M^{\mathsf{T}})$.

Proof. The sum is computed on $\mathcal{E}_+\otimes(\mathcal{X}_2\otimes\mathcal{Y}_2)$; the computation on $\mathcal{E}_-\otimes(\mathcal{X}_2\otimes\mathcal{Y}_2)$ is identical. For a 2×2 array $M = \sum_{j,k}|j\rangle\langle k|\otimes M_{jk}$ of 4×4 blocks, the identities $J|0\rangle = -|1\rangle$ and $J|1\rangle = |0\rangle$ give

$$M + (J\otimes\mathbb{1})M(J\otimes\mathbb{1})^{\mathsf{T}} = \mathbb{1}\otimes(M_{00}+M_{11}) + |0\rangle\langle1|\otimes(M_{01}-M_{10}) + |1\rangle\langle0|\otimes(M_{10}-M_{01}). \tag{20}$$

By the description of the g_i after (13), the restriction of P_0+P_1 to $\mathcal{E}_+\otimes(\mathcal{X}_2\otimes\mathcal{Y}_2)$ is the left side of (20) with $M = P^{++}$, and the restriction of P_2+P_3 is the same expression with M the array P^{--} transported to $\mathcal{E}_+$ by the identity of $\mathbb{C}^2$. Adding the two, the sum equals $\mathbb{1}$ if and only if $\mathrm{Tr}_{\mathcal{E}_+}(P^{++}) + \mathrm{Tr}_{\mathcal{E}_-}(P^{--}) = \mathbb{1}$ and $(P^{++}_{01} - P^{++}_{10}) + (P^{--}_{01} - P^{--}_{10}) = 0$, and since P_0 is symmetric, $P^{\pm\pm}_{10} = (P^{\pm\pm}_{01})^{\mathsf{T}}$, so the second condition is the vanishing of the antisymmetric parts. □

The template

Definition 5. Let $x = (x_0, x_1)$ and $y = (y_0, y_1)$ be vectors in $\mathbb{R}^2$. Define the vector $w \in \mathcal{E}_+ \otimes \mathcal{S}$ and the real numbers

$$w = x_0|00\rangle|00\rangle + y_0|00\rangle|11\rangle + x_1|11\rangle|00\rangle + y_1|11\rangle|11\rangle, \qquad \ell = \frac{x_0y_0 + x_1y_1}{2}, \quad e = \frac{x_0y_1 - x_1y_0}{2}, \quad m = \frac{x_0y_0 - x_1}{2} \tag{21}$$

the 2×2 matrices

$$\overline{W} = \begin{pmatrix} \|x\|^2 & x \cdot y \\ x \cdot y & \|y\|^2 \end{pmatrix}, \qquad L = \tfrac{1}{2}(\mathbb{1} - \overline{W}), \qquad \widetilde{N} = \begin{pmatrix} \ell & e \\ e & -\ell \end{pmatrix}, \qquad \widetilde{N}' = \begin{pmatrix} e & -\ell \\ -\ell & -e \end{pmatrix}, \tag{22}$$

and, with $\rho = \sqrt{\ell^2 + e^2}$ and $\delta = (1 - 4\rho)/4$,

$$M = (\rho + \delta)\mathbb{1} + \frac{m}{\rho}\widetilde{N}, \qquad M' = (\rho + \delta)\mathbb{1} + \frac{m'}{\rho}\widetilde{N}', \qquad G_+ = \begin{pmatrix} M_{00} & M'_{01} \\ M'_{01} & M_{11} \end{pmatrix}, \qquad G_- = \begin{pmatrix} M'_{00} & M_{01} \\ M_{01} & M'_{11} \end{pmatrix}, \tag{23}$$

where $M = M' = \delta\mathbb{1}$ when $\rho = 0$. The operator $P_0(x, y)$ is the real symmetric operator on $\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2$ that is block diagonal for $(\mathcal{E}_+ \oplus \mathcal{E}_-) \otimes (\mathcal{S} \oplus \mathcal{C}_+ \oplus \mathcal{C}_-)$, with blocks

$$|w\rangle\langle w| \text{ on } \mathcal{E}_+ \otimes \mathcal{S}, \qquad \mathbb{1} \otimes L + e\,\sigma_y \otimes \sigma_y \text{ on } \mathcal{E}_- \otimes \mathcal{S}, \qquad G_+ \text{ on } \mathcal{E}_+ \otimes \mathcal{C}_+ \text{ and on } \mathcal{E}_+ \otimes \mathcal{C}_-, \qquad G_- \text{ on } \mathcal{E}_- \otimes \mathcal{C}_+, \tag{24}$$

the $\mathcal{E}_- \otimes \mathcal{S}$ block written in the coordinates of (12), and $P_i(x, y) = (g_i \otimes \mathbb{1})P_0(x, y)(g_i \otimes \mathbb{1})^*$ for $i = 0, 1, 2, 3$.

Two identities underlie the definition. Lagrange's identity $(x_0y_0 + x_1y_1)^2 + (x_0y_1 - x_1y_0)^2 = \|x\|^2\|y\|^2$ gives $\rho = \frac{1}{2}\|x\|\|y\|$, and expanding the squares gives

$$\ell^2 + e^2 = \frac{x_0^2y_0^2 + x_1^2y_1^2 + x_0^2y_1^2 + x_1^2y_0^2}{4} = m^2 + m'^2, \tag{25}$$

so that $|m| \le \rho$ and $|m'| \le \rho$; also $\widetilde{N}^2 = \widetilde{N}'^2 = \rho^2\mathbb{1}$ and $\operatorname{Tr}\widetilde{N} = \operatorname{Tr}\widetilde{N}' = 0$.

Proposition 6. *Suppose that*

$$\|x\|^2 + \|y\|^2 \le 1, \qquad (x_0 + x_1)^2 \le 1, \qquad (y_0 + y_1)^2 \le 1. \tag{26}$$

Then $(P_0(x, y), \ldots, P_3(x, y))$ is a PPT measurement, and its success probability is $F(x, y)^2$, where

$$F(x, y) = \sqrt{\lambda_0}\,(\alpha x_0 + \beta x_1) + \sqrt{\lambda_1}\,(\alpha y_0 + \beta y_1) = \sqrt{\lambda_0}\,(x \cdot a) + \sqrt{\lambda_1}\,(y \cdot a). \tag{27}$$

Proof. The proof checks, in turn, that P_0 is positive semidefinite, that the four operators sum to the identity, that $\mathrm{T}_{\mathcal{Y}}(P_0)$ is positive semidefinite, and the value; the three inequalities (26) are referred to as (T1), (T2) and (T3).

Positivity of P_0. The block $|w\rangle\langle w|$ is positive semidefinite. For the block on $\mathcal{E}_- \otimes \mathcal{S}$, Lemma 3 applies with $K = L$: one has $4\det L = (1 - \|x\|^2)(1 - \|y\|^2) - (x \cdot y)^2$ and, by Lagrange's identity, $4e^2 = \|x\|^2\|y\|^2 - (x \cdot y)^2$, so that

$$4\,(\det L - e^2) = 1 - \|x\|^2 - \|y\|^2 \ge 0 \tag{28}$$

by (T1), while $\operatorname{Tr} L = 1 - \frac{1}{2}(\|x\|^2 + \|y\|^2) \geq 0$; hence $L \succeq 0$, $\det L \geq e^2$, and the block is positive semidefinite. For the charge blocks, $\delta \geq 0$ because $4\rho = 2\|x\|\|y\| \leq \|x\|^2 + \|y\|^2 \leq 1$; since $|m| \leq \rho$, the matrix $\rho\mathbb{1} + \frac{m}{\rho}\widetilde{N}$ has the eigenvalues $\rho \pm m \geq 0$, so $M \succeq \delta\mathbb{1} \succeq 0$, and likewise $M' \succeq \delta\mathbb{1}$. Then $G_+ \succeq 0$, because its diagonal entries are at least δ and, by (25),

$$M_{00}M_{11} - (M'_{01})^2 = (\rho+\delta)^2 - \frac{m^2\ell^2}{\rho^2} - \frac{m'^2\ell^2}{\rho^2} = (\rho+\delta)^2 - \ell^2 \geq \rho^2 - \ell^2 = e^2 \geq 0, \qquad (29)$$

and $G_- \succeq 0$ because $M'_{00}M'_{11} - M_{01}^2 = (\rho+\delta)^2 - e^2(m'^2+m^2)/\rho^2 = (\rho+\delta)^2 - e^2 \geq \ell^2 \geq 0$. If $\rho = 0$ then $\|x\|\|y\| = 0$, the off-diagonal entries of the $\mathcal{S}$ blocks vanish, and $G_\pm = \delta\mathbb{1} = \frac{1}{4}\mathbb{1}$; the same conclusions hold.

Completeness. By Lemma 4 two conditions are to be checked. For the trace condition, sector by sector on $\mathcal{X}_2 \otimes \mathcal{Y}_2$: on $\mathcal{S}$, $\operatorname{Tr}_{\mathcal{E}_+}|w\rangle\langle w| + \operatorname{Tr}_{\mathcal{E}_-}(\mathbb{1}\otimes L + e\,\sigma_y \otimes \sigma_y) = \overline{W} + 2L + e\,\operatorname{Tr}(\sigma_y)\sigma_y = \overline{W} + (\mathbb{1} - \overline{W}) = \mathbb{1}$; on $\mathcal{C}_+$, $\operatorname{Tr} G_+ + \operatorname{Tr} G_- = \operatorname{Tr} M + \operatorname{Tr} M' = 2(\rho+\delta) + 2(\rho+\delta) = 4\rho + 4\delta = 1$; on $\mathcal{C}_-$, $\operatorname{Tr} G_+ + \operatorname{Tr}(JG_-J^\intercal) = 1$; and there are no entries between different sectors. For the antisymmetric condition, the block P_{01}^{++} restricted to $\mathcal{S}$ is the 2×2 array $\big(\langle 00|\langle k|\,w\rangle\big(\langle w|\,|11\rangle|k'\rangle\big)_{kk'} = \left(\begin{smallmatrix} x_0x_1 & x_0y_1 \\ y_0x_1 & y_0y_1 \end{smallmatrix}\right)$, whose antisymmetric part is $e\,(|0\rangle\langle 1| - |1\rangle\langle 0|)$, and on $\mathcal{C}_+$ and $\mathcal{C}_-$ it is the scalar M'_{01}, which is symmetric; the block P_{01}^{--} receives nothing from $\mathbb{1}\otimes L$, receives $e\,\langle 0|\sigma_y|1\rangle\,\sigma_y = -ie\,\sigma_y = e\,(-|0\rangle\langle 1| + |1\rangle\langle 0|)$ on $\mathcal{S}$ from $e\,\sigma_y\otimes\sigma_y$, which is antisymmetric and cancels the previous term, and receives the scalars M_{01} on $\mathcal{C}_+$ and $-M_{01}$ on $\mathcal{C}_-$, both symmetric. Hence (19) holds and $\sum_i P_i = \mathbb{1}$.

Positivity of $\mathrm{T}_{\mathcal{Y}}(P_0)$. By (5), $\mathrm{T}_{\mathcal{Y}_1}$ fixes the entries of an operator that are diagonal in the $\mathcal{E}_+$ or $\mathcal{E}_-$ coordinates and exchanges the off-diagonal entries of $\mathcal{E}_+$ with those of $\mathcal{E}_-$ (it maps $|00\rangle\langle 11|$ to $|01\rangle\langle 10|$ and back), and $\mathrm{T}_{\mathcal{Y}_2}$ fixes the entries diagonal in the $\mathcal{S}$, $\mathcal{C}_+$ or $\mathcal{C}_-$ coordinates and exchanges the off-diagonal entries of $\mathcal{S}$ with the entries between $\mathcal{C}_+$ and $\mathcal{C}_-$ (it maps $|00\rangle\langle 11|$ to $|01\rangle\langle 10|$ on $\mathcal{X}_2\otimes\mathcal{Y}_2$ and back). Applying this to (24), $\mathrm{T}_{\mathcal{Y}}(P_0)$ is block diagonal with four 4×4 blocks:

$$B_1 = \begin{pmatrix} x_0^2 & 0 & 0 & 0 \\ 0 & x_1^2 & 0 & 0 \\ 0 & 0 & L_{00} & x_0x_1 \\ 0 & 0 & x_0x_1 & L_{00} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} y_0^2 & 0 & 0 & 0 \\ 0 & y_1^2 & 0 & 0 \\ 0 & 0 & L_{11} & y_0y_1 \\ 0 & 0 & y_0y_1 & L_{11} \end{pmatrix}, \qquad (30)$$

on $(\mathcal{X}_1\otimes\mathcal{Y}_1)\otimes|00\rangle$ and on $(\mathcal{X}_1\otimes\mathcal{Y}_1)\otimes|11\rangle$ respectively, in the basis $|00\rangle, |11\rangle, |01\rangle, |10\rangle$ of the signal pair (the diagonal entries of the $\mathcal{S}$ blocks stay, and the off-diagonal entries x_0x_1 and y_0y_1 of $|w\rangle\langle w|$ move to $\mathcal{E}_-$), and

$$B_{3a} = \begin{pmatrix} G_{+,00} & G_{-,01} & x_0y_0 & -e \\ G_{-,01} & G_{+,11} & e & x_1y_1 \\ x_0y_0 & e & G_{+,00} & -G_{-,01} \\ -e & x_1y_1 & -G_{-,01} & G_{+,11} \end{pmatrix}, \qquad B_{3b} = \begin{pmatrix} G_{-,00} & G_{+,01} & L_{01} & x_0y_1 \\ G_{+,01} & G_{-,11} & x_1y_0 & L_{01} \\ L_{01} & x_1y_0 & G_{-,11} & G_{+,01} \\ x_0y_1 & L_{01} & G_{+,01} & G_{-,00} \end{pmatrix}, \qquad (31)$$

on $\mathcal{E}_+\otimes(\mathcal{C}_+\oplus\mathcal{C}_-)$ in the basis $|00\rangle|01\rangle$, $|11\rangle|01\rangle$, $|00\rangle|10\rangle$, $|11\rangle|10\rangle$ and on $\mathcal{E}_-\otimes(\mathcal{C}_+\oplus\mathcal{C}_-)$ in the basis $|01\rangle|01\rangle$, $|10\rangle|01\rangle$, $|01\rangle|10\rangle$, $|10\rangle|10\rangle$; here the entries between $\mathcal{C}_+$ and $\mathcal{C}_-$ are the $\mathcal{S}$ off-diagonal entries of $|w\rangle\langle w|$ (x_jy_k), of $\mathbb{1}\otimes L$ (L_{01}) and of $e\,\sigma_y\otimes\sigma_y$ ($\pm e$), moved by $\mathrm{T}_{\mathcal{Y}_2}$, with the $\mathcal{E}_+$ off-diagonal ones moved by $\mathrm{T}_{\mathcal{Y}_1}$ into $\mathcal{E}_-$ and conversely, and the diagonal blocks are the G blocks with their off-diagonal entries exchanged between $\mathcal{E}_+$ and $\mathcal{E}_-$.

Now $B_1 \succeq 0$ if and only if $L_{00} \geq x_0x_1$, that is, $\frac{1}{2}(1 - \|x\|^2) \geq x_0x_1$, that is, $(x_0+x_1)^2 \leq 1$, which is (T2); and $B_2 \succeq 0$ if and only if (T3). For B_{3a}, write it as $\left(\begin{smallmatrix} M & N \\ N^\intercal & \sigma_z M \sigma_z \end{smallmatrix}\right)$ with $N = \left(\begin{smallmatrix} x_0y_0 & -e \\ e & x_1y_1 \end{smallmatrix}\right)$, using $G_{+,00} = M_{00}$, $G_{+,11} = M_{11}$ and $G_{-,01} = M_{01}$. Conjugation by $\mathrm{diag}(\mathbb{1}, \sigma_z)$ turns it into $\left(\begin{smallmatrix} M & N' \\ N' & M \end{smallmatrix}\right)$ with $N' = N\sigma_z = \left(\begin{smallmatrix} x_0 y_0 & e \\ e & -x_1 y_1 \end{smallmatrix}\right) = m\mathbb{1} + \widetilde{N}$, a symmetric matrix, and a further conjugation by $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{smallmatrix}\right)$ turns that into $(M + N') \oplus (M - N')$. The matrix $\rho\mathbb{1} + \frac{m}{\rho}\widetilde{N}$ is positive semidefinite (its eigenvalues are $\rho \pm m$) and squares to $(\rho^2 + m^2)\mathbb{1} + 2m\widetilde{N} = N'^2$, so it is the absolute value $|N'|$, and therefore $M = |N'| + \delta\mathbb{1} \succeq \pm N'$, which proves $B_{3a} \succeq 0$. For B_{3b}, write it as $\left(\begin{smallmatrix} M' & N'' \\ N''^{\mathsf{T}} & \sigma_x M' \sigma_x \end{smallmatrix}\right)$ with $N'' = \left(\begin{smallmatrix} L_{01} & x_0 y_1 \\ x_1 y_0 & L_{01} \end{smallmatrix}\right)$, using $G_{-,00} = M'_{00}$, $G_{-,11} = M'_{11}$ and $G_{+,01} = M'_{01}$; conjugation by $\mathrm{diag}(\mathbb{1}, \sigma_x)$ gives $\left(\begin{smallmatrix} M' & N''' \\ N''' & M' \end{smallmatrix}\right)$ with $N''' = N''\sigma_x = \left(\begin{smallmatrix} x_0 y_1 & L_{01} \\ L_{01} & x_1 y_0 \end{smallmatrix}\right) = m'\mathbb{1} + \widetilde{N}'$, since $L_{01} = -\ell$ and $x_0 y_1 - m' = e = m' - x_1 y_0$, and as before $|N'''| = \rho\mathbb{1} + \frac{m'}{\rho}\widetilde{N}'$ and $M' = |N'''| + \delta\mathbb{1} \succeq \pm N'''$, which proves $B_{3b} \succeq 0$. Since conjugation by $g_i \otimes \mathbb{1}$ preserves the PPT property, every P_i is PPT.

The value. Since ρ_0 is supported on $\mathcal{E}_+ \otimes \mathcal{S}$, where P_0 acts as $|w\rangle\langle w|$, one has $\langle P_0, \rho_0 \rangle = |\langle w| (|\psi_0\rangle \otimes |\tau\rangle)|^2 = F(x,y)^2$ by (21) and (12), and by covariance $\langle P_i, \rho_i \rangle = \langle P_0, \rho_0 \rangle$ for each i, so the success probability (1) is $F(x,y)^2$. □

The best member of the family

Proposition 7. *The maximum of $F(x,y)$ over all (x,y) satisfying (26) equals 1 if $v \le 0$, attained at $x = \sqrt{\lambda_0}\, a$ and $y = \sqrt{\lambda_1}\, a$, and equals $\frac{1}{2}\left(\sqrt{1+v} + \sqrt{1-v}\right)$ if $0 < v < 1$, attained at*

$$x = \left(\frac{1+r}{2}, \frac{1-r}{2}\right), \qquad y = \kappa\, a, \tag{32}$$

with r and κ as in (11). In the second case $\frac{1}{4}\left(\sqrt{1+v} + \sqrt{1-v}\right)^2 = \frac{1}{2}(1+s)$.

Proof. It will first be verified that the two stated choices are feasible and give the stated values, and then that no feasible choice does better.

If $v \le 0$, the choice $x = \sqrt{\lambda_0}\, a$, $y = \sqrt{\lambda_1}\, a$ satisfies $\|x\|^2 + \|y\|^2 = \lambda_0 + \lambda_1 = 1$, $(x_0 + x_1)^2 = \lambda_0(\alpha + \beta)^2 = \lambda_0(1 + c) = 1 + v \le 1$ by (10), and $(y_0 + y_1)^2 = \lambda_1(1 + c) \le \lambda_0(1 + c) \le 1$; its value is $F = \lambda_0\|a\|^2 + \lambda_1\|a\|^2 = 1$. If $0 < v < 1$, the choice (32) satisfies $\|x\|^2 = \frac{1}{2}(1 + r^2) = 1 - \kappa^2$ and $\|y\|^2 = \kappa^2$ by (11), so (T1) holds with equality; $x_0 + x_1 = 1$, so (T2) holds with equality; and $(y_0 + y_1)^2 = \kappa^2(1 + c) = \lambda_1(1 + v)/(\lambda_0(1 - v))$ by (10), which is at most 1 if and only if $\lambda_1(1 + v) \le \lambda_0(1 - v)$, that is, $v(\lambda_0 + \lambda_1) \le \lambda_0 - \lambda_1 = \varepsilon$, which holds. Its value is computed from $x \cdot a = \frac{1}{2}(\alpha + \beta) + \frac{r}{2}(\alpha - \beta)$, $y \cdot a = \kappa$, the last identity of (10) and $\sqrt{\lambda_1}\,\kappa = \lambda_1/\sqrt{1 - v}$:

$$F = \frac{\sqrt{1+v}}{2} + \frac{r\sqrt{\varepsilon - v}}{2} + \frac{\lambda_1}{\sqrt{1-v}} = \frac{\sqrt{1+v}}{2} + \frac{(\varepsilon - v) + 2\lambda_1}{2\sqrt{1-v}} = \frac{\sqrt{1+v} + \sqrt{1-v}}{2}, \tag{33}$$

using $r\sqrt{\varepsilon - v} = (\varepsilon - v)/\sqrt{1 - v}$ and the third identity of (10); squaring gives $\frac{1}{4}(2 + 2\sqrt{1 - v^2}) = \frac{1}{2}(1 + s)$.

For the upper bound, drop (T3), which can only enlarge the feasible set, and put $t = \|x\|$ and $u = \|y\|$, so that $t^2 + u^2 \le 1$. By the Cauchy–Schwarz inequality $y \cdot a \le u$. Let $g(t)$ be the maximum of $x \cdot a$ over $\|x\| = t$ and $(x_0 + x_1)^2 \le 1$. If $t(\alpha + \beta) \le 1$ the unconstrained maximizer $x = ta$ is admissible and $g(t) = t$; if $t(\alpha + \beta) > 1$ the maximizer ta violates (T2), and on the arc $\{\|x\| = t,\ |x_0 + x_1| \le 1\}$, which contains neither ta nor $-ta$ (both have $|x_0 + x_1| = t(\alpha + \beta) > 1$), the function $x \cdot a = t\cos(\theta - \theta_a)$ of the angle θ of x has no critical point on either of the two connected pieces of the arc, so it is monotone on each and attains its maximum at an endpoint, hence on the line $x_0 + x_1 = 1$ (every point of the line $x_0 + x_1 = -1$ is the negative of a point of the line $x_0 + x_1 = 1$, where $x \cdot a$ is positive), where $x = \left(\frac{1+r'}{2}, \frac{1-r'}{2}\right)$ with $r' = \pm\sqrt{2t^2-1}$ and $x \cdot a = \frac{1}{2}(\alpha+\beta) + \frac{r'}{2}(\alpha-\beta)$, maximal for $r' = +\sqrt{2t^2-1}$. Since F increases with u, one has $F \le h(t) = \sqrt{\lambda_0}\, g(t) + \sqrt{\lambda_1}\sqrt{1-t^2}$ for $t \in [0,1]$.

On $[0, 1/(\alpha+\beta)]$, $h(t) = \sqrt{\lambda_0}\, t + \sqrt{\lambda_1}\sqrt{1-t^2} \le 1$ by the Cauchy–Schwarz inequality, with equality if and only if $t = \sqrt{\lambda_0}$. If $v \le 0$ then $\lambda_0(1+c) = 1+v \le 1$, that is, $\sqrt{\lambda_0} \le 1/(\alpha+\beta)$, so the value 1 is attained on this interval, and it is the maximum because $F \le \sqrt{\lambda_0}\, t + \sqrt{\lambda_1}\, u \le \sqrt{\lambda_0+\lambda_1}\sqrt{t^2+u^2} \le 1$ for every feasible choice, by two applications of the Cauchy–Schwarz inequality. If $v > 0$ then $\sqrt{\lambda_0} > 1/(\alpha+\beta)$ and h is increasing on $[0, 1/(\alpha+\beta)]$, its derivative $\sqrt{\lambda_0} - \sqrt{\lambda_1}\, t/\sqrt{1-t^2}$ being positive for $t < \sqrt{\lambda_0}$, so the maximum lies on $[1/(\alpha+\beta), 1]$, where by the last identity of (10)

$$h(t) = \frac{\sqrt{1+v}}{2} + \frac{1}{2}\left[\sqrt{\varepsilon - v}\,\sigma_t + \sqrt{2\lambda_1}\sqrt{1-\sigma_t^2}\right], \qquad \sigma_t = \sqrt{2t^2-1} \in [0,1], \tag{34}$$

since $1 - t^2 = \frac{1}{2}(1-\sigma_t^2)$. By the Cauchy–Schwarz inequality the bracket is at most $\sqrt{(\varepsilon - v) + 2\lambda_1} = \sqrt{1-v}$, with equality if and only if $\sigma_t = \sqrt{(\varepsilon-v)/(1-v)} = r$. Hence the maximum of h is $\frac{1}{2}(\sqrt{1+v} + \sqrt{1-v})$, attained at $2t^2 - 1 = r^2$, that is, $t^2 = 1 - \kappa^2$ and $u^2 = \kappa^2$, which is the choice (32); and that choice satisfies the dropped constraint (T3), so it is the maximum over (26). $\square$

Remark 8. Within the family of Definition 5 the only obstruction to perfect discrimination is the constraint $x_0 + x_1 \le 1$, which is the positivity of the block B_1 of the partial transpose, and the optimum saturates it whenever $v > 0$; the number $v = c\lambda_0 - \lambda_1$ is the excess of $\lambda_0(1+c) = \|\sqrt{\lambda_0}\, a\|^2(\alpha+\beta)^2$ over one, and the remaining optimization is one application of the Cauchy–Schwarz inequality to two vectors of squared norms $1 - v$ and $1 + v$. On the plateau $v \le 0$ the measurement element P_0 contains ρ_0 itself as its $\mathcal{E}_+ \otimes \mathcal{S}$ block, and the other blocks, which are orthogonal to every ρ_i, are what make it complete and PPT.

4 The dual witness

Throughout this section $0 < v < 1$ and $\lambda_1 > 0$ (so $\varepsilon < 1$). The dual operator Y is taken real, in the commutant of G and of the phase rotations, which by the description at the end of Section 2 means $Y = \mathbb{1} \otimes \widetilde{K} - f\,(\sigma_y \otimes \sigma_x) \otimes \widetilde{\sigma_y}$ for a 2×2 block $\widetilde{K}$ supported on $\mathcal{S}$ and $\widetilde{\sigma_y}$ the operator σ_y on $\mathcal{S}$ (in the coordinates of (12)) and zero on $\mathcal{C}_+ \oplus \mathcal{C}_-$; and Q_0 is taken supported on the vector $|\Psi^-\rangle \otimes |00\rangle$, the direction in which the partial transpose of the optimal measurement of Proposition 7 has a zero eigenvalue, since $L_{00} = x_0 x_1$ when $x_0 + x_1 = 1$. Three abbreviations remove most radicals:

$$A = \sqrt{1+v}, \qquad B = \sqrt{1-v}, \qquad D = \sqrt{\varepsilon - v}, \qquad \text{so that} \quad A^2 + B^2 = 2, \quad s = AB, \quad r = \frac{D}{B}, \quad 2\lambda_0 = A^2 + D^2, \tag{35}$$

and by (10), $\sqrt{\lambda_0}\, a = \left(\frac{A+D}{2}, \frac{A-D}{2}\right)$, $\sigma = AD/\lambda_0$, $\sqrt{\lambda_1}\,\kappa = \lambda_1/B$, and $1/\kappa = B/\sqrt{\lambda_1}$.

Definition 9. Put

$$q = \frac{v}{4}, \qquad d = \frac{\lambda_1 v(1+c)}{8s} = \frac{\lambda_1 v A}{8\lambda_0 B}, \qquad c_K = \frac{1+s}{8} - d, \qquad f = \frac{\sqrt{\lambda_0\lambda_1}\, v\sigma}{8s} = \frac{\sqrt{\lambda_1}\, vD}{8\sqrt{\lambda_0}\, B}, \tag{36}$$

$$K = c_K |t\rangle\langle t| + dP_1 = \begin{pmatrix} c_K\lambda_0 & c_K\sqrt{\lambda_0\lambda_1} \\ c_K\sqrt{\lambda_0\lambda_1} & c_K\lambda_1 + d \end{pmatrix}, \qquad K' = K - \frac{q}{2}P_0, \tag{37}$$

and

$$Y = \mathbb{1}_{\mathcal{X}_1\otimes\mathcal{Y}_1} \otimes \widetilde{K} - f\,(\sigma_y \otimes \sigma_x) \otimes \widetilde{\sigma_y}, \qquad Q_0 = q\,|\Psi^-\rangle\langle\Psi^-| \otimes |00\rangle\langle 00|, \qquad Q_i = (g_i \otimes \mathbb{1}) Q_0 (g_i \otimes \mathbb{1})^*, \tag{38}$$

where $\widetilde{K}$ equals K on $\mathcal{S}$ and zero on $\mathcal{C}_+ \oplus \mathcal{C}_-$.

Here $q > 0$, $d > 0$ and $f \geq 0$, with $f = 0$ exactly on the Bell line, and $c_K > 0$ (this is shown in the proof of Proposition 12).

Lemma 10. *The operator Y is real symmetric and commutes with $g_i \otimes \mathbb{1}$ for $i = 0, \ldots, 3$; each Q_i is positive semidefinite; for each i,*

$$Y - \tfrac{1}{4}\rho_i - \mathrm{T}_{\mathcal{Y}}(Q_i) = (g_i \otimes \mathbb{1})\big(Y - \tfrac{1}{4}\rho_0 - \mathrm{T}_{\mathcal{Y}}(Q_0)\big)(g_i \otimes \mathbb{1})^*; \tag{39}$$

the slack for $i = 0$ is $C_+ \oplus C_- \oplus 0$, with 0 the zero operator on $(\mathcal{X}_1 \otimes \mathcal{Y}_1) \otimes (\mathcal{C}_+ \oplus \mathcal{C}_-)$ and, in the coordinates of (12),

$$C_+ = \mathbb{1} \otimes K - f\,\sigma_y \otimes \sigma_y + \frac{q}{2}\,\sigma_x \otimes P_0 - \frac{1}{4}|a\rangle\langle a| \otimes |t\rangle\langle t| \ \text{ on } \mathcal{E}_+ \otimes \mathcal{S}, \qquad C_- = \mathbb{1} \otimes K' - f\,\sigma_y \otimes \sigma_y \ \text{ on } \mathcal{E}_- \otimes \mathcal{S}; \tag{40}$$

and $\mathrm{Tr}(Y) = 4\,\mathrm{Tr}(K) = 4(c_K + d) = \frac{1}{2}(1 + s)$.

Proof. The five assertions are verified in turn. The operators $\sigma_y \otimes \sigma_x$ and $\widetilde{\sigma_y}$ are imaginary and antisymmetric, so their tensor product is real and symmetric, and $\widetilde{K}$ is real symmetric; hence Y is real symmetric, and it commutes with $g \otimes \mathbb{1}$ for $g \in G$ because $\mathbb{1}$ and $\sigma_y \otimes \sigma_x$ span the commutant of G. Each Q_i is q times a projector, so $Q_i \succeq 0$. Equation (39) follows from (14), from $Q_i = (g_i \otimes \mathbb{1})Q_0(g_i \otimes \mathbb{1})^*$, from (6) applied to $g_i \otimes \mathbb{1}$ (whose Bob factor is real) and from $(g_i \otimes \mathbb{1})Y(g_i \otimes \mathbb{1})^* = Y$. For the block form of the slack, ρ_0 is supported on $\mathcal{E}_+ \otimes \mathcal{S}$ where it is $|a\rangle\langle a| \otimes |t\rangle\langle t|$; Y restricted to $\mathcal{E}_+ \otimes \mathcal{S}$ and to $\mathcal{E}_- \otimes \mathcal{S}$ is $\mathbb{1} \otimes K - f\,\sigma_y \otimes \sigma_y$, because $\sigma_y \otimes \sigma_x$ acts as σ_y on both sectors, and Y vanishes on the charge sectors; and by Lemma 2, $\mathrm{T}_{\mathcal{Y}}(Q_0) = q\ \mathrm{T}_{\mathcal{Y}_1}(|\Psi^-\rangle\langle\Psi^-|) \otimes \mathrm{T}_{\mathcal{Y}_2}(|00\rangle\langle 00|) = q\left(\frac{1}{2}\mathbb{1} - |\Phi^+\rangle\langle\Phi^+|\right) \otimes |00\rangle\langle 00|$, which is $-\frac{q}{2}\sigma_x \otimes P_0$ on $\mathcal{E}_+ \otimes \mathcal{S}$, $\frac{q}{2}\mathbb{1} \otimes P_0$ on $\mathcal{E}_- \otimes \mathcal{S}$, and zero on the charge sectors. Subtracting gives (40). Finally $\mathrm{Tr}(Y) = \mathrm{Tr}(\mathbb{1}_{\mathcal{X}_1\otimes\mathcal{Y}_1})\,\mathrm{Tr}(\widetilde{K}) - f\,\mathrm{Tr}(\sigma_y \otimes \sigma_x)\,\mathrm{Tr}(\widetilde{\sigma_y}) = 4\,\mathrm{Tr}(K)$, and $\mathrm{Tr}(K) = c_K(\lambda_0 + \lambda_1) + d = \frac{1}{8}(1 + s)$. □

The kernel of the slack

Lemma 11. *Let $x = \left(\frac{1+r}{2}, \frac{1-r}{2}\right)$ be the first component of the optimizer (32), let $Zx = (x_0, -x_1)$, and in the array convention of Section 2 define*

$$\widetilde{w} = \left[\kappa^{-1}x,\ a\right], \qquad w_2 = \left[\kappa^{-1}Zx,\ -a'\right]. \tag{41}$$

Then $C_+\widetilde{w} = 0$ and $C_+w_2 = 0$.

Proof. The proof rationalizes C_+ by a diagonal congruence and then displays the two products as polynomials in A, B and D that vanish because $A^2 + B^2 = 2$.

In the basis $|00\rangle|00\rangle$, $|00\rangle|11\rangle$, $|11\rangle|00\rangle$, $|11\rangle|11\rangle$ of $\mathcal{E}_+ \otimes \mathcal{S}$, with $K_{00} = c_K\lambda_0$, $K_{01} = c_K\sqrt{\lambda_0\lambda_1}$, $K_{11} = c_K\lambda_1 + d$, and $\sigma_y \otimes \sigma_y = -(|00\rangle\langle 11| + |11\rangle\langle 00|) + (|01\rangle\langle 10| + |10\rangle\langle 01|)$ in that basis, the definition (40) reads

$$C_+ = \begin{pmatrix} K_{00} - \frac{\lambda_0\alpha^2}{4} & K_{01} - \frac{\sqrt{\lambda_0\lambda_1}\,\alpha^2}{4} & \frac{q}{2} - \frac{\lambda_0\alpha\beta}{4} & f - \frac{\sqrt{\lambda_0\lambda_1}\,\alpha\beta}{4} \\ \cdot & K_{11} - \frac{\lambda_1\alpha^2}{4} & -f - \frac{\sqrt{\lambda_0\lambda_1}\,\alpha\beta}{4} & -\frac{\lambda_1\alpha\beta}{4} \\ \cdot & \cdot & K_{00} - \frac{\lambda_0\beta^2}{4} & K_{01} - \frac{\sqrt{\lambda_0\lambda_1}\,\beta^2}{4} \\ \cdot & \cdot & \cdot & K_{11} - \frac{\lambda_1\beta^2}{4} \end{pmatrix}, \tag{42}$$

a symmetric matrix whose entries below the diagonal are omitted. Let $\Lambda = \operatorname{diag}(\kappa, \sqrt{\lambda_0})$ on $\mathcal{S}$ and $\widehat{C} = (\mathbb{1}\otimes\Lambda)^{-1}C_+(\mathbb{1}\otimes\Lambda)^{-1}$: an entry of C_+ whose $\mathcal{S}$ indices are (k,k') is multiplied by B^2/λ_1, by $B/\sqrt{\lambda_0\lambda_1}$, or by $1/\lambda_0$ according as $(k,k') = (0,0)$, $(0,1)$ or $(1,1)$, since $\kappa^2 = \lambda_1/B^2$. Since $C_+u = 0$ if and only if $\widehat{C}(\mathbb{1}\otimes\Lambda)u = 0$, it suffices to prove $\widehat{C}\widehat{w} = 0$ and $\widehat{C}\widehat{w}_2 = 0$ for

$$\widehat{w} = (\mathbb{1}\otimes\Lambda)\widetilde{w} = \Big(\frac{B+D}{2B}, \frac{A+D}{2}, \frac{B-D}{2B}, \frac{A-D}{2}\Big), \qquad \widehat{w}_2 = (\mathbb{1}\otimes\Lambda)w_2 = \Big(\frac{B+D}{2B}, -\frac{A-D}{2}, -\frac{B-D}{2B}, \frac{A+D}{2} \tag{43}$$

written as flat vectors in the basis above, where $\kappa\cdot\kappa^{-1} = 1$, $x_0 = \frac{1}{2}(1+D/B)$, $x_1 = \frac{1}{2}(1-D/B)$ and $\sqrt{\lambda_0}\,a = (\frac{A+D}{2}, \frac{A-D}{2})$ were used.

Substituting (36) and (35) into (42), together with $\alpha^2 = \frac{(A+D)^2}{4\lambda_0}$, $\beta^2 = \frac{(A-D)^2}{4\lambda_0}$, $\alpha\beta = \frac{A^2-D^2}{4\lambda_0}$ and $v = A^2 - 1$, every $\sqrt{\lambda_0}$ and $\sqrt{\lambda_1}$ cancels against the scaling, and one finds

$$16\,\widehat{C} = \begin{pmatrix} \dfrac{AB(B-D)^2}{\lambda_1} & \dfrac{A(B-D)^2}{\lambda_0} & -2B^2 & -\dfrac{B(A^2-D^2)-2Dv}{\lambda_0} \\ \cdot & \dfrac{2A\lambda_1(\lambda_0-BD)}{B\lambda_0^2} & -\dfrac{B(A^2-D^2)+2Dv}{\lambda_0} & -\dfrac{\lambda_1(A^2-D^2)}{\lambda_0^2} \\ \cdot & \cdot & \dfrac{AB(B+D)^2}{\lambda_1} & \dfrac{A(B+D)^2}{\lambda_0} \\ \cdot & \cdot & \cdot & \dfrac{2A\lambda_1(\lambda_0+BD)}{B\lambda_0^2} \end{pmatrix}. \tag{44}$$

The four entries carrying a factor A^2-D^2 or B^2 are immediate: $16(\frac{q}{2} - \frac{\lambda_0\alpha\beta}{4})\frac{B^2}{\lambda_1} = (2v - (A^2 - D^2))\frac{B^2}{\lambda_1} = (A^2+D^2-2)\frac{B^2}{\lambda_1} = -2B^2$, because $A^2+D^2-2 = 2\lambda_0 - 2 = -2\lambda_1$; $16(f - \frac{\sqrt{\lambda_0\lambda_1}\,\alpha\beta}{4})\frac{B}{\sqrt{\lambda_0\lambda_1}} = \frac{2vD}{\lambda_0} - \frac{B(A^2-D^2)}{\lambda_0}$; $16(-f - \frac{\sqrt{\lambda_0\lambda_1}\,\alpha\beta}{4})\frac{B}{\sqrt{\lambda_0\lambda_1}} = -\frac{2vD}{\lambda_0} - \frac{B(A^2-D^2)}{\lambda_0}$; and $16(-\frac{\lambda_1\alpha\beta}{4})\frac{1}{\lambda_0} = -\frac{\lambda_1(A^2-D^2)}{\lambda_0^2}$. The remaining six rest on one polynomial identity: with $2\lambda_0 = A^2+D^2$, $2\lambda_1 = B^2-D^2$ and $v = 1-B^2$,

$$2(1+AB)\lambda_0 B - 2\lambda_1 vA - B(A\pm D)^2 = A(B\mp D)^2 + AD^2(A^2+B^2-2) = A(B\mp D)^2, \tag{45}$$

Expanding the left side, the terms $B(A^2+D^2)$ and $B(A^2\pm 2AD + D^2)$ cancel up to $\mp 2ABD$ and what remains is

$$AB^2(A^2+D^2) - A(B^2-D^2)(1-B^2) \mp 2ABD = A\big[B^2(A^2+B^2-1) + D^2 \mp 2BD\big], \tag{46}$$

and $A^2+B^2-1 = 1$ gives $A(B\mp D)^2$. Then $16\widehat{C}_{00,00} = \big[16c_K\lambda_0 - (A+D)^2\big]\frac{B^2}{\lambda_1} = \big[2(1+AB)\lambda_0 - \frac{2\lambda_1 vA}{B} - (A+D)^2\big]\frac{B^2}{\lambda_1} = \frac{AB(B-D)^2}{\lambda_1}$ by (45) with the upper signs, and $16\widehat{C}_{10,10}$ is the same with D replaced by $-D$; $16\widehat{C}_{00,01} = (16c_K - 4\alpha^2)B = \big[2(1+AB)\lambda_0 B - 2\lambda_1 vA - B(A+D)^2\big]/\lambda_0 = A(B-D)^2/\lambda_0$, and $16\widehat{C}_{10,11}$ likewise with D replaced by $-D$; and $16\widehat{C}_{01,01} = \big[16c_K\lambda_1 + 16d -$

$4\lambda_1\alpha^2\big]/\lambda_0 = \lambda_1\big[16c_K - 4\alpha^2 + \frac{2vA}{\lambda_0 B}\big]/\lambda_0 = \frac{\lambda_1 A[(B-D)^2+2v]}{\lambda_0^2 B} = \frac{\lambda_1 A[(A^2+D^2)-2BD]}{\lambda_0^2 B} = \frac{2\lambda_1 A(\lambda_0 - BD)}{\lambda_0^2 B}$, using $(B-D)^2 + 2v = B^2 - 2BD + D^2 + 2A^2 - 2 = A^2 + D^2 - 2BD$, with $16\widehat{C}_{11,11}$ likewise.

Multiplying (44) by (43) and clearing the denominators, the four components of $16\widehat{C}\widehat{w}$ are

$$
\begin{aligned}
(A^2+D^2)\,(16\widehat{C}\widehat{w})_0 &= A^3D - A^2D^2 + AB^2D - 2AD - B^2D^2 + 2D^2 = D(A-D)(A^2+B^2-2),\\
B(A^2+D^2)^2\,(16\widehat{C}\widehat{w})_1 &= -A^4BD + A^4D^2 - A^2B^3D + A^2B^2D^2 - A^2BD^3 + 2A^2BD + A^2D^4 - 2A^2D^2 - B^3D^3 + B\\
&= -D(A^2+D^2)(B-D)(A^2+B^2-2),\\
(A^2+D^2)\,(16\widehat{C}\widehat{w})_2 &= -A^3D - A^2D^2 - AB^2D + 2AD - B^2D^2 + 2D^2 = -D(A+D)(A^2+B^2-2),\\
B(A^2+D^2)^2\,(16\widehat{C}\widehat{w})_3 &= A^4BD + A^4D^2 + A^2B^3D + A^2B^2D^2 + A^2BD^3 - 2A^2BD + A^2D^4 - 2A^2D^2 + B^3D^3 + B^2L\\
&= D(A^2+D^2)(B+D)(A^2+B^2-2),
\end{aligned}
\tag{47}
$$

and the four components of $16\widehat{C}\widehat{w}_2$, multiplied by the same denominators, are $D(A+D)(A^2+B^2-2)$, $D(A^2+D^2)(B-D)(A^2+B^2-2)$, $D(A-D)(A^2+B^2-2)$ and $D(A^2+D^2)(B+D)(A^2+B^2-2)$ respectively. Each factorization is verified by multiplying out its right side. Since $A^2+B^2=2$, all eight components vanish, which completes the proof. □

The parameters (36) are not a guess: expanding $C_+\widetilde{w} = 0$ in the basis $\{a, a'\}$ of $\mathbb{R}^2$ gives four equations that are linear in f, q, c_K and d with the vector x of (32) as data, and (36) is their unique solution, after which $C_+ w_2 = 0$ holds automatically. In other words the dual witness is determined by complementary slackness with the measurement of Proposition 7, and the proof above verifies the result of that determination.

Feasibility

Proposition 12. *For $0 < v < 1$ and $\lambda_1 > 0$, the operators C_- and C_+ of (40) are positive semidefinite. Consequently $(Y, Q_0, \ldots, Q_3)$ is feasible for the dual problem and* $\mathrm{opt}(\alpha,\varepsilon) \le \mathrm{Tr}(Y) = \frac{1}{2}(1+s)$.

Proof. It will first be proved that $C_- \succeq 0$, by Lemma 3, and then that $C_+ \succeq 0$, by a rank argument.

The block C_-. By Lemma 3 applied to K' it suffices to show $\mathrm{Tr}\,K' \ge 0$ and $\det K' \ge f^2$. First, $\mathrm{Tr}\,K' = c_K + d - \frac{q}{2} = \frac{1+s}{8} - \frac{v}{8} = \frac{1+s-v}{8} > 0$. Second, with $K'_{00} = c_K\lambda_0 - \frac{v}{8}$, $K'_{01} = c_K\sqrt{\lambda_0\lambda_1}$ and $K'_{11} = c_K\lambda_1 + d$,

$$
\det K' = \Big(c_K\lambda_0 - \frac{v}{8}\Big)(c_K\lambda_1 + d) - c_K^2\lambda_0\lambda_1 = d\Big(c_K\lambda_0 - \frac{v}{8}\Big) - \frac{vc_K\lambda_1}{8},
\tag{48}
$$

and inserting $c_K = \frac{1+s}{8} - d$ gives $\det K' = \frac{d\lambda_0(1+s-v)}{8} - \lambda_0 d^2 - \frac{v\lambda_1(1+s)}{64}$. Now $d = \frac{\lambda_1 v(1+v)}{8\lambda_0 s}$ by (10), and $s^2 = (1+v)(1-v)$, so

$$
\frac{d\lambda_0(1+s-v)}{8} = \frac{\lambda_1 v(1+v)(1+s-v)}{64s}, \qquad \lambda_0 d^2 = \frac{\lambda_1^2 v^2(1+v)}{64\lambda_0(1-v)}, \qquad f^2 = \frac{\lambda_0\lambda_1 v^2\sigma^2}{64s^2} = \frac{\lambda_1 v^2(\varepsilon - v)}{64\lambda_0(1-v)},
\tag{49}
$$

the last because $\lambda_0\sigma^2 = \lambda_0(1-c)(1+c) = (\varepsilon - v)(1+v)/\lambda_0$ by (10). Therefore

$$
\det K' - f^2 = \frac{\lambda_1 v}{64}\Big\{\frac{(1+v)(1+s-v)}{s} - \frac{v\big[\lambda_1(1+v) + (\varepsilon - v)\big]}{\lambda_0(1-v)} - (1+s)\Big\},
\tag{50}
$$

and $\lambda_1(1+v)+(\varepsilon-v)=\lambda_1+\varepsilon+v(\lambda_1-1)=\lambda_0-v\lambda_0=\lambda_0(1-v)$, so the middle term is v and the brace equals $\frac{(1+v)(1+s-v)-s(1+v+s)}{s}=\frac{1-v^2-s^2}{s}=0$. Hence $\det K'=f^2$, and $C_-\succeq 0$ with eigenvalues $\frac{1+s-v}{8}$ (twice) and 0 (twice). In particular $K'\succeq 0$ forces $K'_{00}=c_K\lambda_0-\frac{v}{8}\geq 0$, so $c_K>0$ as claimed after Definition 9.

The block C_+. C_+ is a real symmetric 4×4 matrix which, by Lemma 11, annihilates $\widetilde{w}$ and w_2; these are linearly independent, since their $\mathcal{S}$-components a and $-a'$ are orthogonal and nonzero. Hence the kernel of C_+ has dimension at least two and the characteristic polynomial of C_+ is $\mu^2(\mu^2-e_1\mu+e_2)$ with $e_1=\operatorname{Tr}C_+$ and $e_2=\frac{1}{2}\big[(\operatorname{Tr}C_+)^2-\operatorname{Tr}(C_+^2)\big]$, by Newton's identity for the two remaining eigenvalues. The terms $\sigma_y\otimes\sigma_y$ and $\sigma_x\otimes P_0$ are traceless, so $e_1=2\operatorname{Tr}K-\frac{1}{4}\|a\|^2\|t\|^2=\frac{1+s}{4}-\frac{1}{4}=\frac{s}{4}$. For $\operatorname{Tr}(C_+^2)$ write $C_+=\Omega+\frac{q}{2}\sigma_x\otimes P_0-\frac{1}{4}|at\rangle\langle at|$ with $\Omega=\mathbb{1}\otimes K-f\sigma_y\otimes\sigma_y$ and $|at\rangle=a\otimes t$ a unit vector; then $\operatorname{Tr}(\Omega^2)=2\operatorname{Tr}(K^2)+4f^2$ (the cross term carries $\operatorname{Tr}\sigma_y=0$), $\operatorname{Tr}\big((\sigma_x\otimes P_0)^2\big)=2$, $\operatorname{Tr}\big(\Omega(\sigma_x\otimes P_0)\big)=0$, $\langle at|\Omega|at\rangle=\langle t|K|t\rangle=c_K+d\lambda_1$ (as $\langle a|\sigma_y|a\rangle=0$ for real a) and $\langle at|\sigma_x\otimes P_0|at\rangle=c\lambda_0$, so that

$$\operatorname{Tr}(C_+^2)=2\big(c_K^2+2c_Kd\lambda_1+d^2\big)+4f^2+\frac{q^2}{2}+\frac{1}{16}-\frac{c_K+d\lambda_1}{2}-\frac{qc\lambda_0}{4},\tag{51}$$

using $\operatorname{Tr}(K^2)=c_K^2(\lambda_0+\lambda_1)^2+2c_Kd\lambda_1+d^2$. Substituting $c_K=\frac{1+s}{8}-d$, $q=\frac{v}{4}$, $c\lambda_0=v+\lambda_1$ and $s^2+v^2=1$, the constant terms collapse, $\frac{(1+s)^2}{32}+\frac{v^2}{32}+\frac{1}{16}-\frac{1+s}{16}=\frac{1+s^2+v^2}{32}=\frac{1}{16}$, and what remains is

$$\operatorname{Tr}(C_+^2)=\frac{1}{16}-\frac{v^2}{16}-\frac{v\lambda_1}{16}-\frac{sd\lambda_0}{2}+4\lambda_0d^2+4f^2.\tag{52}$$

Here $\frac{sd\lambda_0}{2}=\frac{\lambda_1 v(1+v)}{16}$, and $4\lambda_0d^2+4f^2=\frac{\lambda_1v^2[\lambda_1(1+v)+(\varepsilon-v)]}{16\lambda_0(1-v)}=\frac{\lambda_1v^2}{16}$ by the same bracket as above, so that $\operatorname{Tr}(C_+^2)=\frac{1-v^2}{16}-\frac{v\lambda_1}{16}-\frac{\lambda_1v(1+v)}{16}+\frac{\lambda_1v^2}{16}=\frac{s^2}{16}-\frac{\lambda_1v}{8}$, and $e_2=\frac{1}{2}\big[\frac{s^2}{16}-\frac{s^2}{16}+\frac{\lambda_1v}{8}\big]=\frac{\lambda_1v}{16}\geq 0$. The two eigenvalues of C_+ that may be nonzero are real, their sum is $\frac{s}{4}>0$ and their product is $\frac{\lambda_1v}{16}\geq 0$, so both are nonnegative and $C_+\succeq 0$.

By Lemma 10, every slack $Y-\frac{1}{4}\rho_i-\mathrm{T}_{\mathcal{Y}}(Q_i)$ is unitarily conjugate to $C_+\oplus C_-\oplus 0\succeq 0$ and every $Q_i\succeq 0$, so the pair is dual feasible, and Proposition 1 gives $\operatorname{opt}(\alpha,\varepsilon)\leq\operatorname{Tr}(Y)=\frac{1}{2}(1+s)$. □

Remark 13. The nonzero eigenvalues of C_+ are $\frac{1}{8}\big(s\pm\sqrt{s^2-4\lambda_1v}\big)$, the roots of $\mu^2-\frac{s}{4}\mu+\frac{\lambda_1v}{16}$, and those of C_- are $\frac{1}{8}(1+s-v)$, twice. The kernel of the full slack has dimension twelve, and complementary slackness with the measurement of Proposition 7 holds: its element P_0 is supported in that kernel, and $\mathrm{T}_{\mathcal{Y}}(P_0)$ annihilates $|\Psi^-\rangle\otimes|00\rangle$.

5 The optimal PPT success probability

The two halves are now combined. The sides of the parameter square on which the general certificates degenerate (where s, λ_1 or $1-v$ vanish) are covered by the elementary measurements and one-line dual solutions recorded in the proof.

Theorem 14. *Let $\mathcal{X}_1=\mathcal{X}_2=\mathcal{Y}_1=\mathcal{Y}_2=\mathbb{C}^2$, let $\alpha\geq\beta\geq 0$ with $\alpha^2+\beta^2=1$ and $\varepsilon\in[0,1]$ be chosen arbitrarily, let $|\psi_0\rangle,\ldots,|\psi_3\rangle$ be the states* (4) *and $|\tau_\varepsilon\rangle$ the state* (3)*, and let*

$$v=2\alpha\beta\,\frac{1+\varepsilon}{2}-\frac{1-\varepsilon}{2},\qquad v_+=\max\{v,0\}.\tag{53}$$

The optimal probability of correctly discriminating the states $|\psi_i\rangle \otimes |\tau_\varepsilon\rangle$, $i = 0, 1, 2, 3$, given with uniform probabilities, by a PPT measurement with respect to the bipartition $(\mathcal{X}_1 \otimes \mathcal{X}_2) : (\mathcal{Y}_1 \otimes \mathcal{Y}_2)$, is

$$\operatorname{opt}(\alpha, \varepsilon) = \frac{1}{2}\Big(1 + \sqrt{1 - v_+^2}\Big). \tag{54}$$

In particular the four states are perfectly discriminated by a PPT measurement if and only if $2\alpha\beta(1+\varepsilon) \le 1 - \varepsilon$.

Proof. The proof splits the closed square into four cases according to the values of v and λ_1.

If $v \le 0$, Proposition 6 and Proposition 7 give a PPT measurement with success probability 1, and 1 is an upper bound for any measurement, so $\operatorname{opt}(\alpha, \varepsilon) = 1$, in agreement with (54).

If $0 < v < 1$ and $\lambda_1 > 0$, Propositions 6 and 7 give $\operatorname{opt}(\alpha, \varepsilon) \ge \frac{1}{2}(1 + s)$ and Proposition 12 gives $\operatorname{opt}(\alpha, \varepsilon) \le \frac{1}{2}(1 + s)$.

If $\lambda_1 = 0$, that is $\varepsilon = 1$, then $|\tau_1\rangle = |00\rangle$ and $v = c$. The projectors $P_0 = |00\rangle\langle 00| \otimes \mathbb{1}$, $P_1 = |11\rangle\langle 11| \otimes \mathbb{1}$, $P_2 = |01\rangle\langle 01| \otimes \mathbb{1}$ and $P_3 = |10\rangle\langle 10| \otimes \mathbb{1}$ (a product projector on the signal pair tensored with the identity on the resource pair) form a PPT measurement: they are positive, they sum to $\mathbb{1}$, and each is diagonal in the computational basis of $\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2$, hence fixed by $\mathrm{T}_{\mathcal{Y}}$ by (5); their success probability is $\frac{1}{4}(\alpha^2 + \alpha^2 + \alpha^2 + \alpha^2) = \alpha^2$, since each $|\psi_i\rangle$ has weight α^2 on its assigned product vector. For the matching dual witness put

$$Y = \frac{\alpha^2}{4} \mathbb{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1} \otimes |00\rangle\langle 00|, \qquad Q_i = \mathrm{T}_{\mathcal{Y}}(Y) - \frac{1}{4}\mathrm{T}_{\mathcal{Y}}(\rho_i) = \frac{1}{4}\Big[\alpha^2 \mathbb{1} - \mathrm{T}_{\mathcal{Y}_1}\big(|\psi_i\rangle\langle\psi_i|\big)\Big] \otimes |00\rangle\langle 00|; \tag{55}$$

by Lemma 2 and (14) the eigenvalues of $\mathrm{T}_{\mathcal{Y}_1}(|\psi_i\rangle\langle\psi_i|)$ are α^2, β^2 and $\pm\alpha\beta$, all at most α^2, so $Q_i \succeq 0$, while $\mathrm{T}_{\mathcal{Y}}(Y) = Y$ and $\mathrm{T}_{\mathcal{Y}}$ is an involution, so $Y - \frac{1}{4}\rho_i - \mathrm{T}_{\mathcal{Y}}(Q_i) = 0$. Thus $\operatorname{opt}(\alpha, 1) = \operatorname{Tr}(Y) = \alpha^2$, and $\alpha^2 = \frac{1}{2}(1 + \sigma) = \frac{1}{2}(1 + \sqrt{1 - c^2})$ is (54). (If $c = 0$ this case also falls under $v = 0$.)

If $v = 1$, then $\lambda_1 = 0$ and $c = 1$ by the remarks after (10), the single point $\alpha = \beta$, $\varepsilon = 1$, which is inside the previous case and gives $\operatorname{opt} = \frac{1}{2}$; it is listed separately only because $s = 0$ there.

The four cases cover the closed square, and the last assertion follows because $\sqrt{1 - v_+^2} < 1$ exactly when $v > 0$. □

Remark 15. The hypothesis $\alpha \ge \beta$ is inessential: exchanging α and β in (4) and applying the local unitary $Z \otimes \mathbb{1}$ returns the same four states in the order $\psi_1, \psi_0, \psi_3, \psi_2$, and (54) depends on the Schmidt coefficients only through $c = 2\alpha\beta$. The uniform prior is essential: with the prior $(1, 0, 0, 0)$ the measurement $P_0 = \mathbb{1}$ is PPT and succeeds with certainty at every (α, ε). On the boundary curve $v = 0$ the two measurements of Proposition 7 coincide, since there $r = \sqrt{\varepsilon}$, $\kappa = \sqrt{\lambda_1}$ and $x = \sqrt{\lambda_0}\, a$, and as $\lambda_1 \to 0$ the parameters (36) tend to $d = f = 0$ and $c_K = \alpha^2/4$, so that Y tends to the dual witness (55).

Corollary 16. *The following values are recovered from (54), and each was also obtained directly. For $\varepsilon = 0$, $v = \frac{1}{2}(c - 1) \le 0$ and $\operatorname{opt} = 1$, the value of the teleportation protocol through the maximally entangled resource. For $\alpha = 1$, $v = -\lambda_1 \le 0$ and $\operatorname{opt} = 1$, the value of the measurement in the product basis. For $\alpha = \beta$, $v = \varepsilon$ and $\operatorname{opt} = \frac{1}{2}(1 + \sqrt{1 - \varepsilon^2})$, in agreement with Theorem 5 of [BCJ+14]; and for $\alpha = \beta$, $\varepsilon = 1$, $\operatorname{opt} = \frac{1}{2}$, the value for four Bell states without a resource. On the Bell line the region of perfect discrimination reduces to $\varepsilon = 0$, in agreement with [YDY12].*

Example

Let $\alpha = 2/\sqrt{5}$, $\beta = 1/\sqrt{5}$ and $\varepsilon = 7/9$, so that $c = 4/5$, $\sigma = 3/5$, $\lambda_0 = 8/9$, $\lambda_1 = 1/9$, $v = \frac{4}{5}\cdot\frac{8}{9} - \frac{1}{9} = \frac{3}{5}$ and $s = \frac{4}{5}$. Theorem 14 gives $\mathrm{opt}(2/\sqrt{5}, 7/9) = \frac{1}{2}\big(1+\frac{4}{5}\big) = \frac{9}{10}$. Here $r = 2/3$ and $\kappa = \sqrt{5/18}$, so the measurement of Proposition 7 is built from $x = (5/6, 1/6)$ and $y = (\sqrt{2}/3, \sqrt{2}/6)$, and its success probability is $F^2 = \big(\sqrt{8/9}\cdot\frac{11}{6\sqrt{5}} + \sqrt{1/9}\cdot\sqrt{5/18}\big)^2 = \big(\frac{11\sqrt{2}}{9\sqrt{5}} + \frac{\sqrt{10}}{18}\big)^2 = \big(\frac{3\sqrt{10}}{10}\big)^2 = \frac{9}{10}$; the dual witness of Definition 9 has

$$q = \frac{3}{20}, \qquad d = \frac{3}{160}, \qquad c_K = \frac{33}{160}, \qquad f = \frac{\sqrt{2}}{80}, \qquad K = \begin{pmatrix} 11/60 & 11\sqrt{2}/240 \\ 11\sqrt{2}/240 & 1/24 \end{pmatrix}, \qquad \mathrm{Tr}(Y) = 4\,\mathrm{Tr}(K) = \tag{56}$$

and the nonzero eigenvalues of the slack are $\frac{1}{8}\big(\frac{4}{5} \pm \sqrt{\frac{28}{75}}\big)$ on $\mathcal{E}_+ \otimes \mathcal{S}$ and $\frac{1}{8}\cdot\frac{6}{5}$ on $\mathcal{E}_- \otimes \mathcal{S}$. The teleportation protocol of Proposition 17 below succeeds with probability $\frac{1}{2}\big(1+\sqrt{1-(28/45)^2}\big) = \frac{1}{2}\big(1+\frac{\sqrt{1241}}{45}\big) < 0.892$, below $\frac{9}{10}$.

The teleportation protocol

Proposition 17. *Let Alice measure $\mathcal{X}_1 \otimes \mathcal{X}_2$ in the Bell basis $\{|\Phi_k\rangle = (\sigma_k \otimes \mathbb{1})|\Phi^+\rangle\}_{k=0}^3$, with $\sigma_0 = \mathbb{1}$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, and let Bob then perform, for each outcome k, the measurement on $\mathcal{Y}_1 \otimes \mathcal{Y}_2$ that maximizes the probability of identifying i. The success probability of this LOCC protocol is $\frac{1}{2}\big(1+\sqrt{1-(c\varepsilon)^2}\big)$.*

Proof. The conditional states are computed first, and the optimal two-state measurements second. Since $|\Phi_k\rangle = (\sigma_k \otimes \mathbb{1})|\Phi^+\rangle$, one has $\langle\Phi_k|\,|x_1\rangle|x_2\rangle = \frac{1}{\sqrt{2}}\langle x_2|\sigma_k^*|x_1\rangle$ for computational basis vectors of $\mathcal{X}_1 \otimes \mathcal{X}_2$, so Bob's unnormalized conditional vector for outcome k and state i, $|b_{k,i}\rangle = (\langle\Phi_k| \otimes \mathbb{1})(|\psi_i\rangle \otimes |\tau\rangle)$, is obtained from the expansion $|\psi_i\rangle \otimes |\tau\rangle = \sum_{x_1,x_2} a^{(i)}_{x_1} t_{x_2} |x_1\rangle|y_i(x_1)\rangle|x_2\rangle|x_2\rangle$, in which $a^{(0)} = a^{(2)} = a$, $a^{(1)} = a^{(3)} = a'$, $t = (\sqrt{\lambda_0}, \sqrt{\lambda_1})$, $y_i(x_1) = x_1$ for $i \in \{0,1\}$ and $y_i(x_1) = 1 - x_1$ for $i \in \{2,3\}$, as

$$|b_{k,i}\rangle = \frac{1}{\sqrt{2}} \sum_{x_1,x_2 \in \{0,1\}} a^{(i)}_{x_1}\, t_{x_2}\, \langle x_2|\sigma_k^*|x_1\rangle\; |y_i(x_1)\rangle_{\mathcal{Y}_1} |x_2\rangle_{\mathcal{Y}_2}. \tag{57}$$

For $k = 0$ and $k = 3$ the matrix element forces $x_2 = x_1$ and contributes the signs 1 and $(-1)^{x_1}$, and for $k = 1$ and $k = 2$ it forces $x_2 = 1 - x_1$ and contributes 1 and $\pm i$; explicitly, up to the overall phases just mentioned,

$$\begin{aligned}
k \in \{0,3\}: \quad & |b_{k,0}\rangle = \tfrac{1}{\sqrt{2}}\big(\alpha\sqrt{\lambda_0}|00\rangle \pm \beta\sqrt{\lambda_1}|11\rangle\big), \quad |b_{k,1}\rangle = \tfrac{1}{\sqrt{2}}\big(\beta\sqrt{\lambda_0}|00\rangle \mp \alpha\sqrt{\lambda_1}|11\rangle\big), \\
& |b_{k,2}\rangle = \tfrac{1}{\sqrt{2}}\big(\alpha\sqrt{\lambda_0}|10\rangle \pm \beta\sqrt{\lambda_1}|01\rangle\big), \quad |b_{k,3}\rangle = \tfrac{1}{\sqrt{2}}\big(\beta\sqrt{\lambda_0}|10\rangle \mp \alpha\sqrt{\lambda_1}|01\rangle\big), \\
k \in \{1,2\}: \quad & |b_{k,0}\rangle = \tfrac{1}{\sqrt{2}}\big(\alpha\sqrt{\lambda_1}|01\rangle \pm \beta\sqrt{\lambda_0}|10\rangle\big), \quad |b_{k,1}\rangle = \tfrac{1}{\sqrt{2}}\big(\beta\sqrt{\lambda_1}|01\rangle \mp \alpha\sqrt{\lambda_0}|10\rangle\big), \\
& |b_{k,2}\rangle = \tfrac{1}{\sqrt{2}}\big(\alpha\sqrt{\lambda_1}|11\rangle \pm \beta\sqrt{\lambda_0}|00\rangle\big), \quad |b_{k,3}\rangle = \tfrac{1}{\sqrt{2}}\big(\beta\sqrt{\lambda_1}|11\rangle \mp \alpha\sqrt{\lambda_0}|00\rangle\big),
\end{aligned} \tag{58}$$

on $\mathcal{Y}_1 \otimes \mathcal{Y}_2$, with the upper signs for $k \in \{0,1\}$ and the lower signs for $k \in \{2,3\}$. For every k the vectors $b_{k,0}, b_{k,1}$ lie in one of the subspaces $\mathrm{span}\{|00\rangle, |11\rangle\}$, $\mathrm{span}\{|01\rangle, |10\rangle\}$ and $b_{k,2}, b_{k,3}$

in the other, so for each outcome Bob's task splits into two two-state problems on orthogonal subspaces; and for every k and both pairs,

$$\|b_{k,0}\|^2 + \|b_{k,1}\|^2 = \tfrac{1}{2}\big(\alpha^2\lambda_0 + \beta^2\lambda_1 + \beta^2\lambda_0 + \alpha^2\lambda_1\big) = \tfrac{1}{2}, \qquad |\langle b_{k,0}, b_{k,1}\rangle| = \tfrac{1}{2}\alpha\beta\,|\lambda_0 - \lambda_1| = \tfrac{c\varepsilon}{4}, \tag{59}$$

the same two numbers arising for $k \in \{1,2\}$ with λ_0 and λ_1 exchanged, and likewise for the pair $(b_{k,2}, b_{k,3})$. For two vectors b, b' on a two-dimensional space, the largest value of $\langle b|B|b\rangle + \langle b'|(\mathbb{1} - B)|b'\rangle$ over $0 \preceq B \preceq \mathbb{1}$ is $\|b'\|^2 + \mathrm{Tr}\big(|b\rangle\langle b| - |b'\rangle\langle b'|\big)_+ = \frac{1}{2}\big(\|b\|^2 + \|b'\|^2 + \big\||b\rangle\langle b| - |b'\rangle\langle b'|\big\|_1\big)$, and the two eigenvalues of $|b\rangle\langle b| - |b'\rangle\langle b'|$ have sum $\|b\|^2 - \|b'\|^2$ and product $-(\|b\|^2\|b'\|^2 - |\langle b, b'\rangle|^2)$, so its trace norm is $\sqrt{(\|b\|^2 + \|b'\|^2)^2 - 4|\langle b, b'\rangle|^2}$. With (59) each pair contributes $\frac{1}{2}\big(\frac{1}{2} + \frac{1}{2}\sqrt{1 - c^2\varepsilon^2}\big) = \frac{1}{4}\big(1 + \sqrt{1 - c^2\varepsilon^2}\big)$ to $\sum_i \langle b_{k,i}|B_{k,i}|b_{k,i}\rangle$, and summing over the two pairs and the four outcomes and multiplying by the prior $\frac{1}{4}$ gives $\frac{1}{4}\cdot 4\cdot 2\cdot \frac{1}{4}\big(1 + \sqrt{1 - c^2\varepsilon^2}\big) = \frac{1}{2}\big(1 + \sqrt{1 - c^2\varepsilon^2}\big)$. □

Remark 18. Since $c\varepsilon - v = \lambda_1(1 - c)$, the value of Proposition 17 is strictly below (54) whenever $0 < \beta < \alpha$ and $0 < \varepsilon < 1$ (then $\lambda_1(1 - c) > 0$ and $c\varepsilon > 0$), and equal to it on the Bell line, at $\alpha = 1$ and at $\varepsilon \in \{0, 1\}$. The measurement of Definition 5 is therefore not this protocol; whether the value (54) is attained by any separable or LOCC measurement when $\alpha \neq \beta$ is an open question (Section 6).

6 Conclusion and Future Work

In this paper we have used semidefinite programming duality, and matching primal and dual solutions in closed form in particular, to determine the optimal probability with which PPT measurements discriminate a twisted Bell basis when the two parties share a partially entangled pair of qubits, and to characterize the parameters for which the discrimination is perfect.

Several interesting questions regarding this problem remain unsolved. Among them are the following three questions.

- In Section 5 we proved that PPT measurements attain the value $\frac{1}{2}\big(1 + \sqrt{1 - v_+^2}\big)$, and in Proposition 17 that the teleportation protocol attains only $\frac{1}{2}\big(1 + \sqrt{1 - (c\varepsilon)^2}\big)$. Whether separable measurements, or LOCC measurements, attain the PPT value when $\alpha \neq \beta$ is open; the obstruction is that the optimal measurement exhibited here has no product structure, so that no protocol attaining the value is in hand, and no separable dual witness is exhibited. For three Bell states [BCJ+14, YDY12] showed that the separable and PPT values differ, so both answers are possible a priori.

- The bases considered in [BHN17], in which the pairs (ψ_0, ψ_1) and (ψ_2, ψ_3) carry independent Schmidt coefficients (α, β) and (γ, δ), are not covered by Theorem 14, whose symmetry group requires the two pairs to be related by a local unitary. One could ask for the PPT value of such a basis with the resource (3), and more generally for the entanglement cost of discriminating a given basis of two qubits by PPT measurements.

- For three of the four states (4), the PPT value is unknown even on the Bell line for $\varepsilon > 1/3$, where [YDY12] shows that perfect discrimination fails; the three-state problem lacks the transitive symmetry used here, and a different construction of the measurement is required.

The techniques presented in the paper are not intrinsically limited to two qubits with a qubit resource; applications of these techniques to bases of higher local dimension and to resources of higher Schmidt rank are topics for possible future work.

Availability

Artifacts are available from the authors on request.

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